Journal of Statistical Physics, Vol. 114, Nos. 3/4, February 2004 (© 2004)

Discrete Velocity Models and One Continuous Relation

H. Cornille[1]

Received October 24, 2002; accepted June 19, 2003

Some years ago, Cercignani, in studying the energy to mass ratio in a homogeneous equilibrium state, concluded that the associated DVMs (Discrete Velocity Models) ratio would be a drawback for DVMs. Here with planar DVMs, dimension $d=2$, we try to answer this criticism. First, we study four elementary classes of pth squares planar DVMs with or without rest-particle, where the Hausdorff dimensions of the associated lattices, when p increases up to infinity, are $d_H = 0, 1, 2, 2$. The DVM energy to mass ratio leads to *a fictitious dimension* d_p *and the problem is to see whether* $d_p \simeq 2$, *with p increasing or not*. Our result, taking into account the constraints due to the DVM conservation laws, is that this is possible only for the models with $d_H = 2$ for the associated lattices with an infinite number of velocities. We also discuss intermediate cases, for instance p finite for the standard DVMs is sufficient in restricted cases. Second, we study two families of intermediate models between the above $d_H = 1, 2$ with a rest-particle. Only for one family, called α-cross models with $d_H = 2$, do we still find that the continuous mass ratio condition for the dimension can be satisfied.

KEY WORDS: Kinetic theory; discrete models; Boltzmann equation; Hausdorff dimension.

1. INTRODUCTION AND RESULTS FOR SIX PHYSICAL CLASSES OF MODELS

The first connection between DVMs and continuous relations (called the first continuous relation) was in the pioneering Broadwell[1] DVMs infinite shock-wave solution. The continuous mass ratio $\delta_M = M^{(i)}/M^{(ii)}$ relation, at the asymptotic states, was automatically satisfied. Previously[2] we

[1] Service de Physique théorique, CE Saclay, F-91191 Gif-sur-Yvette, France; Unité associée CNRS/SPM/URA 2306; e-mail: cornille@spht.saclay.cea.fr

verified this continuous relation for other infinite shocks, and we also found more general solutions, with constraints on the parameters.

More recently (the second continuous relation), Cercignani criticized (in dimension $d = 3$) the associated results with DVMs.[3, 4] We repeat some of Cercignani's criticisms[5] concerning the standard mass M, energy E, and the E/M ratio:

"In a (homogeneous) equilibrium state we should have

$$M = \int \sum_i M_i \, dx, \qquad E = (1/2) \int \sum_i c_i^2 M_i \, dx, \tag{1a}$$

where M_i is a Maxwellian distribution given by

$$M_i = A \exp(-\beta m c_i^2/2), \qquad \beta = 1/(kT) \tag{1b}$$

and $c_i = v_i$ if we assume the gas at rest... . This gives not only the desired identification of the temperature... satisfactory for the usual Boltzmann gas with continuous velocities, but in the case of a general gas it would imply

$$d = 3, \quad d/(2\beta m) = E/M = (1/2) \sum_i c_i^2 \exp(-\beta m c_i^2/2) / \sum_i \exp(-\beta m c_i^2/2), \tag{1c}$$

i.e., for any value of the parameter b

$$d = 3, \quad d/2b = \sum_i c_i^2 \exp(-b c_i^2) / \sum_i \exp(-b c_i^2), \tag{1d}$$

which is obviously absurd, unless we go to the limit of infinitely many velocities with a continuous distribution."

Our problem is whether the associated planar DVMs can have dimension d equivalent to 2 or not. In the continuous theory, b is inversely proportional to the temperature T. A great subjacent problem is whether we can speak about temperature in DVMs. Since the paper of Cercignani in 1993, many people (myself included) avoid using the word "temperature" in DVMs.

Here we will be mainly interested in whether we can recover the notion of the true continuous dimension in DVMs or not. Our strategy is deduced from that criticism. We recall that the standard DVMs,[1–4] existing for fifty years, have, in general, only a finite number of velocities with many different distributions in the plane. So, the first answer to this criticism is *to see whether the associated DVMs ratios can satisfy d equivalent to 2 or not with a finite number of velocities, here squares.* To a negative answer, we augment the number of squares, eventually up to infinity. The second question, if we can satisfy this criticism, is *to see whether this is possible only for particular distributions of the velocities in the plane or not*. Then the crucial point is to characterize the models satisfying or not satisfying this criticism.

For these reasons we study four elementary models in Fig. 1 and two more complex models in Fig. 3. In fact, with our DVMs a finite number of pth squares, in general, is sufficient to obtain a stable DVM dimension,

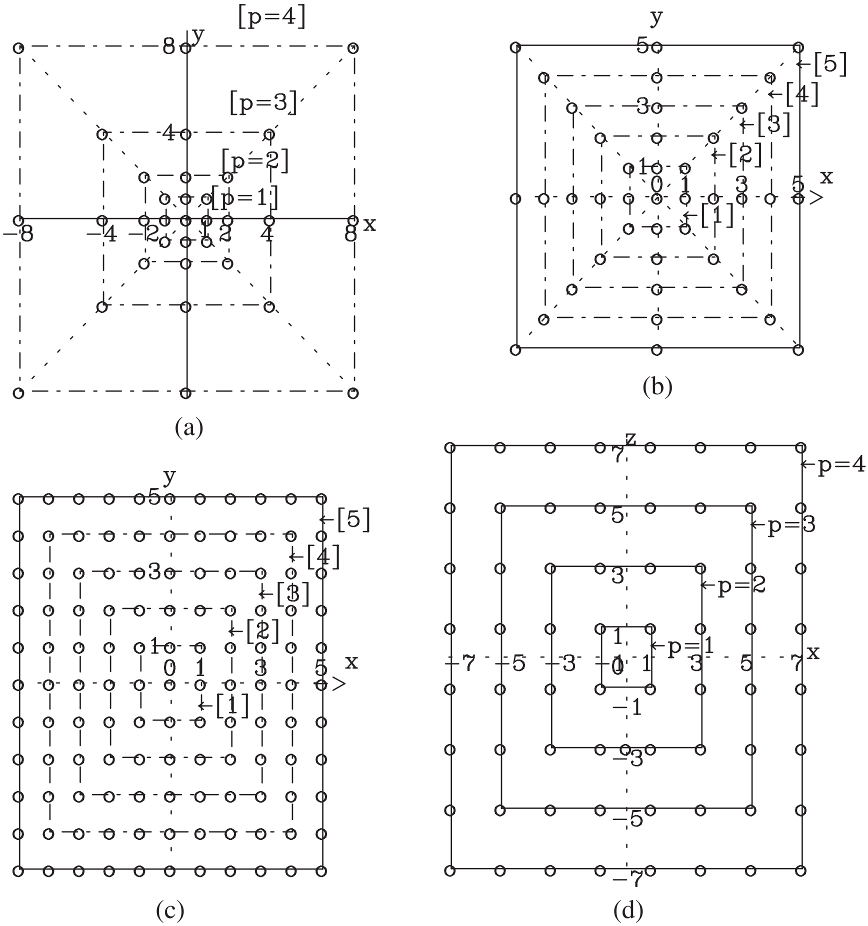

Fig. 1. (a) Nested $p = 1, 2, 3, 4$ squares, (b) Class I filling x, y, $y = \pm x$ axes, $p = 1, 2, \ldots$ 5th squares, (c) Class II filling all coordinates, $p = 1, 2 \ldots 5$th squares, (d) $p = 1, 2, 3, 4$ Nicodin squares.

when b in (1d) is not too small. However, this number is not constant and depends on the values DVMs dimension, but this number is not constant and depends on a parameter similar to b in (1d). An infinite number is necessary only when this parameter goes to zero. It can happen that increasing the number of velocities does not improve our DVM dimension, which is fixed by only a few velocities. This means that in order to recover the planar dimension we must introduce other constraints than (1d).

All our DVMs with finite or infinite p (Section 2) are physical[(6–8)] with only mass, momentum, and energy invariants (Lemmas 1, 1′, and 1″). The tools[(6–8)] are that, for a single-gas (a species for a mixture), starting from a collision with velocities along rectangles or squares and three belonging to a previous physical model, we can add the last one.

First, we study four elementary classes of pth square DVMs, here p arbitrary, with (Figs. 1a–c) or without (Fig. 1d) rest-particle. The Nested[(2)] and Class I[(8)] models (Figs. 1a, b) have velocities $\vec{v}_i$ along the two axes and the bisectors of the plane, but with a uniform repartition only for Class I, as a 1-dimensional lattice when p increases up to infinity. For Class II[(8)] (Fig. 1c), when the number of squares increases up to infinity all integer coordinates of the plane are filled, as a 2-dimensional lattice. The Nicodin models[(9)] without rest-particle (Fig. 1d) fill all odd-integer coordinates of the plane, as a 2-dimensional lattice.

For these four models the associated Hausdorff dimensions (Section 2) are $d_H = 0, 1, 2, 2$. With only Rankine–Hugoniot relations[(2–4)] (no exact solutions), we study one-dimensional traveling waves along the $\vec{x}$-axis with isotropic downstream (i) state and, at the upstream (ii) state, nonzero densities fixed.

Second, we study two classes of models intermediate between Class I and II, where we add to Class I velocities parallel to the $\vec{x}$ and $\vec{y}$ axes. The first, called Extended Class I, and the second, Restricted Class II (Fig. 3), with $d_H = 1, 2$ have either an infinite or a finite number of velocities less than Class II.

In Section 2, we consider the second (1d) Maxwellian associated DVMs relation with only the equilibrium DVMs densities at the (i) isotropic state. Assuming one density normalized to 1, with binary collisions, we obtain all densities as functions of another density. Then we can include for the densities (Eqs. 4c, c′, c″) different terms of the type $e^{-b\vec{v}_q^2}$ (velocity $\vec{v}_q$) and b a continuous parameter. For a pth square, we define a fictitious dimension d_p:

$$p \text{ fixed:} \qquad 2E^{(i)}/M^{(i)} = d_p/2b. \tag{1e}$$

If, for b fixed, we augment the number of squares, then the *fictitious dimension increases (Lemmas 2, 2′, 2″) and becomes stable $d_{p_{\max}}$ for p larger*

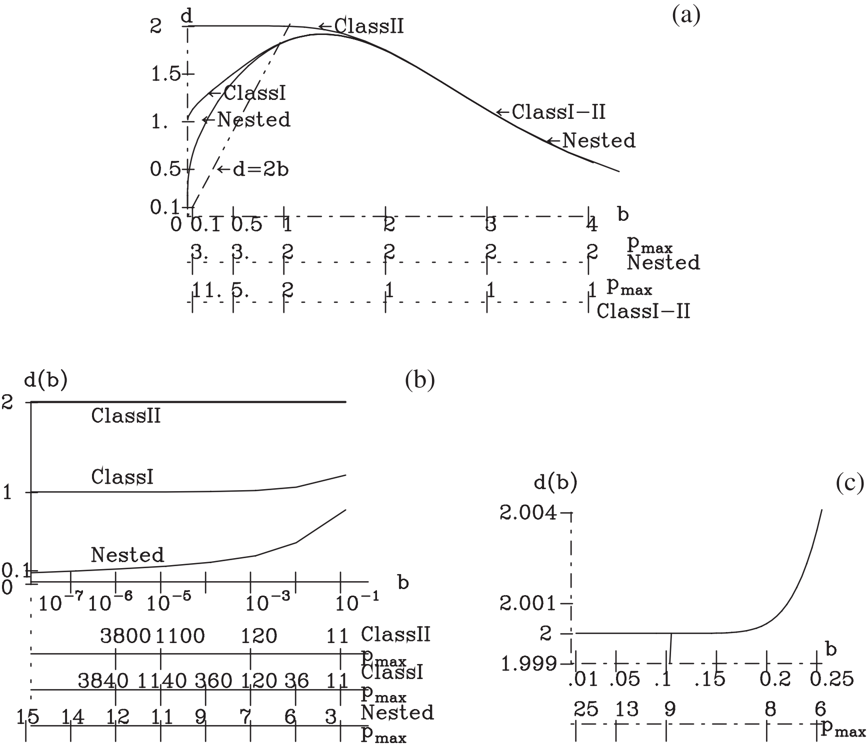

Fig. 2. (a) second continuous: $0.1 < b < 4$, (i) isotropic, rest-particle, Nested $\subset$ Class I $\subset$ Class II, (b) Second continuous: $b = 10^{-r}$, $r = 1, 2, \ldots 8$ rest-particle, Nested $\subset$ Class I $\subset$ Class II, (c) second continuous: $0.1 < b < 0.25$, without rest-particle, Nicodin model.

than $p_{\max}$. But this necessary number of squares is not constant: a finite number for b finite and an infinite number when b decreases up to 0. If we recall that the standard DVMs[(1−3)] mean a fixed number of velocities (here squares), we see that we cannot study the second relation (1e) in this way. Furthermore, the infinite number when b goes to 0 means that we must make a comparison with the associated lattices.

With $p_{\max}$ *varying with* b, *we compare* $d_{p_{\max}}$ *with the planar dimension* $d = 2$. The results are catastrophic (Fig. 2) and justify Cercignani's criticism of DVMs. $d_{p_{\max}}$ is not a constant close to 2 ($\simeq 2$), except for Class II, Nicodin models and only for a fixed b interval $b \in (0, b_{\max})$ including 0. For these models, outside these intervals, either $d_{p_{\max}}$ decreases and is less than 2 or increases with values higher than 2. Increasing the number of velocities does not improve our search of the $d = 2$ dimension and we must introduce other constraints than (1e). For the other two Nested and Class I models, then d different from 2 decreases to 0 or 1 when b decreases to 0. All these results illustrate the weakness of DVMs compared to the continuous theory because the *continuous (1d) relation can be obtained with the (1e) DVMs, only for particular classes of models and with b only in particular intervals.*

Let us go back to the standard DVMs with a *fixed number p of squares* (Section 2.5) for the Nicodin and Class II models. We have the planar dimension d equivalent to 2 only in a finite b interval and the interval increases when the fixed number of squares increases.

In Section 3, always for the second DVMs (1e) relation at the (i) isotropic state, *we add the three conservation laws* linking the (i) state to the (ii) upstream state. Then we get restrictions coming from the positivity of $E^{(i)}/M^{(i)}$ and the *positivity of the discrete microscopic densities* (studied mainly in Appendix B.3). The main result is that b must belong to a well-defined finite interval (Lemmas 3, 3′). Then (Fig. 2) *the Cercignani planar dimension d equivalent to 2 is satisfied for Class II and Nicodin models with* $d_H = 2$ *but not for Nested and Class I models with* $d_H = 0, 1$. However, b in DVMs is not, in general, a continuous parameter. We briefly recall results for infinite-shock solutions at the (ii) state, for Class II models with d equivalent to 2 satisfied from the (i) state.

In Section 4, we study two intermediate classes between the Class I and II models. For the first class (Section 4.1), called Extended Class I, we have $d_H = 1$ and an infinite number of missing Class II velocities, when p increases to infinity. When b goes to to 0, the fictitious dimension cannot satisfy the planar $d = 2$ dimension. For the second (Fig. 3) family (Section 4.2), called α-cross, with only a finite number $4\alpha(\alpha - 1)$ of missing Class II velocities $\vec{v}$, we have $d_H = 2$ and the fictitious dimension can satisfy d equivalent to 2. The *continuous condition for the dimension can be satisfied but with the number p increasing to infinity.*

Finally, in Section 5, we show that if instead of regular grids with mesh steps $= 1$, we choose $1/2, 1/4, \ldots$, the geometrical structure of the curves and the link with the dimension are not changed.

2. FOUR CLASSES OF PHYSICAL DVMS *p*th SQUARES, *p* GOING TO INFINITY

We study four classes of planar DVMs p-squares giving lattices when p goes to infinity: Nested and Class I and II squares with rest-particle $\vec{v}(0, 0)$ (Sections 2.1 and 2.2) and Nicodin squares without rest-particle (Section 2.3). For p fixed, Eqs. (3a, a′, a″), we give the size L_p, the number

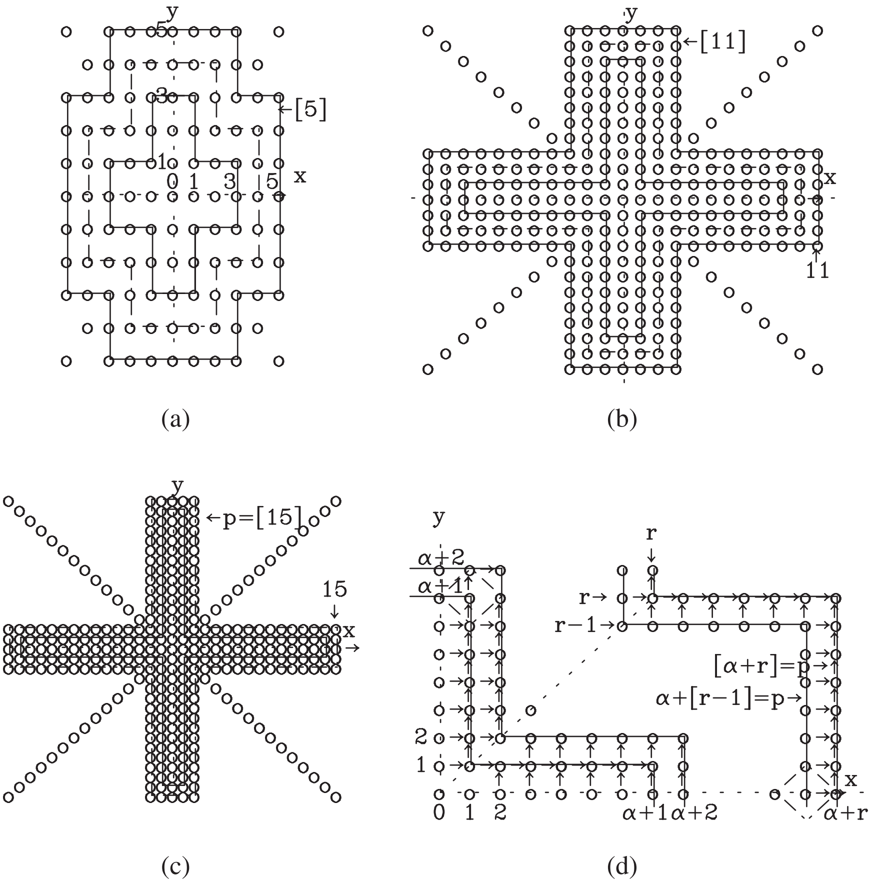

Fig. 3. (a) $\alpha = 2$-Cross, $r = 3$, $p = [5] = [\alpha + r]$, (b) $\alpha = 8$-Cross, $r = 3$, $p = [11] = [\alpha + r]$, (c) $\alpha = 13$-Cross, $r = 2$, $p = [15] = [\alpha + r]$, (d) $p = \alpha + 1$, from $p = \alpha$ physical and α-cross, $p = [\alpha + r]$ from $[\alpha + r - 1]$.

of velocities $\vec{v}_i$ or densities N_p, and the Hausdorff dimension d_H when p goes to infinity.

	N_p	L_p	$d_H = (\log N_p / \log L_p)_{p \to \infty}$
Class I	$N_I = 8p + 1$	$2p$	$d_H \to 1$
Nested	$8p + 1$	2^p	$d_H \to 0$
Class II	$N_{II} = 4p(p+1) + 1$	$2p$	$d_H \to 2$
Nicodin	$4p^2$	$2(2p+1)$	$d_H \to 2$

.

For these planar $d=2$ models, only Class II and Nicodin models have $d_H=2$.

First we explain briefly that these models are physical,[6–8] without spurious invariants. Let us consider *only binary collisions* (no multiple) with four velocities $\vec{v}_i(x, y)$ satisfying

$$\sum_1^2 \vec{v}_i = \sum_3^4 \vec{v}_i, \qquad \sum_1^2 (\vec{v}_i)^2 = \sum_3^4 (\vec{v}_i)^2 \tag{2}$$

and that are at the tops of squares or rectangles. With three velocities belonging to an old physical model, we can add, for a new physical model, the density associated to the last velocity. Starting with an old physical model with only physical conservation relations (no spurious), adding the new collision, we must add the new density in order to satisfy the new conservation relations. *In particular,[6] for a single-gas (such as here) or for a species of a mixture model with four velocities along a square or a rectangle and three densities belonging to a physical model, we can add the last one for a new physical model* (the most important tool here).

Second, we determine the densities associated to an isotropic (i) state and write the second fictitious continuous DVMs relation (1e).

2.1. Class I and Nested *p*-Squares (Figs. 1a, b): $\vec{v}_i(x, y)$ Velocities, *x*, *y* Integers or 0:

$$\vec{v}_i\colon\ (0,0),\quad (\pm a_q, 0),\quad (0, \pm a_q),\quad (\pm a_q, \pm a_q),\quad (\pm a_q, \mp a_q),$$
$$q=1,2,\ldots\, p \text{ integer},\quad \text{Class I } a_q=q,\quad \text{Nested } a_q=2^{q-1} \tag{3a}$$

We see that these models only have velocities along the two coordinate axes and the bisectors, the difference being a uniform distribution only for Class I. First we show that these models are physical, using the recalled tool of binary collisions with three densities known and starting with a physical model. Second, for the isotropic (i) state, we write the associated fictitious (1e) DVMs energy to mass ratio $d_p/2b$. Third, for b fixed, we prove that the fictitious dimension d_p increases with p.

Lemma 1 (Physical Class II[8] and Nested[2] Models). We assume that the pth models are physical and prove that the $p+1$ ones are also physical. Starting with the $p=1$, $9\vec{v}_i$ physical[3,4] model, the $p=2, 3, \ldots$ th models are physical. For the proof we write some velocities $\vec{v}(x, y)$ associated to binary collisions satisfying (2):

Class I:[8] $a_p = p \geqslant 1$

$$\begin{aligned} &\text{(i)} \quad (p, 1)+(p, -1) = (p-1, 0)+(p+1, 0), \\ &\qquad (1, p)+(-1, p) = (0, p-1)+(0, p+1), \\ &\text{(ii)} \quad (p+1, 0)+(0, p+1) = (0, 0)+(p+1, p+1); \end{aligned} \tag{3b}$$

Nested:[2] $a_p = 2^{p-1} = 2a_{p-1} \geqslant 1$

$$\begin{aligned} &\text{(i)} \quad (a_p, a_p)+(a_p, -a_p) = (0, 0)+(a_{p+1}, 0), \\ &\text{(ii)} \quad (a_{p+1}, 0)+(0, a_{p+1}) = (0, 0)+(a_{p+1}, a_{p+1}). \end{aligned} \tag{3c}$$

With (i) we add the new densities associated to the velocity $(a_{p+1}, 0)$ along the $\vec{x}$ axis and $(0, a_{p+1})$ along the $\vec{y}$ axis with x, y symmetry (exchanging x and y in the velocities). From these new densities along the $\vec{x}$, $\vec{y}$ axes and the rest-particle with (ii), we add the new density (a_{p+1}, a_{p+1}) along the bisector. With x, y symmetries we add the associated densities in the other quadrants. Starting with the $p = 1$, $9\vec{v}_i$ model, the $p = 2, 3,...$ and all pth models are physical.

Isotropic (i) State

To the $\vec{v}(x = a_q, y = a_s)$, we associate the densities $n^{(i)}_{a_q, a_s}$ at the isotropic (i) state. All $n^{(i)}_{x, y}$ with the same velocity modulus $|\vec{v}(x, y)|$ are equal, giving a multiplicity of 4 for both $n^{(i)}_{a_q, 0} = n^{(i)}_{0, a_q}$ and $n^{(i)}_{a_q, a_q}$. Keeping only the independent $n^{(i)}_{x, y}$ along the $x \geqslant 0$, $y = x$ axes and the rest-particle $r^{(i)}$, we can write both the mass $M^{(i)}$, energy $E^{(i)}$, and momentum $J^{(i)}$:

$$M^{(i)} = r^{(i)} + \sum_{q=1}^{p} 4[n^{(i)}_{a_q, 0} + n^{(i)}_{a_q, a_q}], \qquad E^{(i)} = \sum_{q} 2a_q^2 [n^{(i)}_{a_q, 0} + 2n^{(i)}_{a_q, a_q}], \quad J^{(i)} = 0. \tag{4a}$$

First, assuming one density normalized to 1, another to $m > 1$, with binary collisions[2–8] we deduce all densities satisfying a general result $n^{(i)}_{x, y} = m^2 / m^{\vec{v}(x, y)^2}$, checked with collisions. Second, we define a *parameter* $b := \log m > 0$, deduce all densities functions of $e^{-ba_q^2}$, and substitute into the macroscopic quantities (4a):

$$\begin{aligned} &1 = a_1 = n^{(i)}_{a_1, a_1}, \qquad n^{(i)}_{0.a_1} = n^{(i)}_{a_1, 0} = m, \qquad n^{(i)}_{x, y} = m^2 / m^{\vec{v}(x, y)^2} = m^{2 - x^2 - y^2}, \\ &n^{(i)}_{a_q, 0} = n^{(i)}_{0, a_q} = m^2 / m^{a_q^2}, \qquad (0, 0) + (1, 1) = (0, 1) + (1, 0), \qquad r^{(i)} = m^2, \end{aligned}$$

$$(0,0)+(a_q,a_q)=(a_q,0)+(0,a_q), \qquad n^{(i)}_{a_q,a_q}=m^2/m^{2a_q^2},$$
$$n^{(i)}_{a_q,a_q}r^{(i)}=n^{(i)}_{0,a_q}n^{(i)}_{a_q,0}=m^4/m^{2a_q^2}... \tag{4b}$$

$$n^{(i)}_{x,y}/m^2=e^{-b\vec{v}(x,y)^2}, \qquad M^{(i)}/4m^2=1/4+\sum_{q=1}^{p} e^{-ba_q^2}(1+e^{-ba_q^2})=\Delta_p=1/4+B_p,$$
$$E^{(i)}/2m^2=\sum_{q=1}^{p} a_q^2 e^{-ba_q^2}(1+2e^{-ba_q^2})=-\partial_b\Delta_p=A_p>A_{p-1}. \tag{4c1}$$

The DVMs energy to mass ratio, written with sums of gaussians $e^{-ba_q^2}$, depends only on b and p, and we write the *second DVMs (1e) relation, with a fictitious dimension d_p:*

$$2E^{(i)}/M^{(i)}=-\partial_b\Delta_p/\Delta_p=A_p/(1/4+B_p)=d_p/2b. \tag{4c2}$$

Lemma 2. For b fixed, d_p increases with p or $d_p \geqslant d_{p-1}$ and it is sufficient to prove $A_pB_{p-1}>A_{p-1}B_p$ (proof with microscopic positivity given in Appendix A.1).

2.2. Class II p-Squares (Fig. 1c), $a_q=q$, $\vec{v}_i(x,y)$ Velocities with x, y Integers or 0:

To Class I, we add all integer coordinates between the $\vec{x}$, $\vec{y}$ axes and the bisectors:

$$\text{Class II: } \vec{v}_i\text{: } (0,0), (\pm q, \pm s), (\pm s, \pm q), (\mp q, \pm s), (\mp s, \pm q),$$
$$s=0,1\ldots q, \quad q=1\ldots p \tag{3a$'$}$$

Lemma 1′ (Physical Class II Models[8]). The $p=1, 9\vec{v}_i$ model is physical.[2–4] From a physical pth square, we prove that the $p+1$ one is physical. We still use binary collisions in squares and rectangles with three associated densities belonging to a previous physical model. We only write (x, y) for $\vec{v}(x, y)$ and use the x, y symmetries for the other densities.

As in (i) and (ii) for Class I (3c), we add the densities associated to $(p+1, 0)$, $(0, p+1)$, and $(p+1, p+1)$ along the two coordinate axes and the bisectors.

In (iii), we add new densities associated to $(p+1, j)$ and $(j, p+1)$, not present in Class I: first the densities parallel to the y-axis with $x=p+1$ and, exchanging x and y, those parallel to the x-axis with $y=p+1$:

$$\text{(iii)} \quad (p+1,0)+(p,j)=(p,0)+(p+1,j), \qquad j=1,2,\ldots,p,$$
$$(0,p+1)+(j,p)=(0,p)+(j,p+1), \qquad j=1,2,\ldots,p. \tag{3b$'$}$$

The $(p+1)$th square is physical and all pth squares are physical.

Isotropic (i) State

The densities associated to velocities with the same modulus being equal, to the Class I densities we only add the new densities parallel to the coordinate axes with a multiplicity of 8 for $n^{(i)}_{q,s}$, $s=1,..., q-1$. Starting with densities $n^{(i)}_{x>0,\, y\geqslant 0}$, we use the x, y symmetries for the other:

$$M^{(i)}=r^{(i)}+\sum_{q=1}^{p} 4\left[n^{(i)}_{q,0}+n^{(i)}_{q,q}+2\sum_{s=1}^{q-1} n^{(i)}_{q,s}\right], \qquad J^{(i)}=0,$$
$$E^{(i)}=\sum_{q}\left[2q^2(n^{(i)}_{q,0}+2n^{(i)}_{q,q})+4\sum_{s=1}^{q-1}(q^2+s^2)\, n^{(i)}_{q,s}\right]. \tag{4a$'$}$$

With the assumptions of Class I, we deduce $n^{(i)}_{q,s}$ and the second DVMs continuous relation:

$$n^{(i)}_{1,1}=1, \qquad n^{(i)}_{0,1}=m, \qquad b=\log m, \qquad n^{(i)}_{q,s}/m^2=e^{-b(q^2+s^2)},$$
$$\Delta_p=M^{(i)}/4m^2=1/4+B_p, \qquad E^{(i)}/2m^2=-\partial_b\Delta_p=A_p,$$
$$B_p=\sum_{q=1}^{p}\left[e^{-bq^2}(1+e^{-bq^2})+2\sum_{s=1}^{q-1}e^{-b(q^2+s^2)}\right]=1/4+B_p, \tag{4b$'$}$$
$$A_p=\sum_{q=1}^{p}\left[e^{-bq^2}q^2(1+2e^{-bq^2})+2\sum_{s=1}^{q-1}(q^2+s^2)\, e^{-b(q^2+s^2)}\right],$$
$$2E^{(i)}/M^{(i)}=d_p/2b=-\partial_b\Delta_p/\Delta_p=A_p/(1/4+B_p).$$

Lemma 2$'$. For b fixed, d_p increases with p or $d_p\geqslant d_{p-1}$ and it is sufficient to prove $A_pB_{p-1}>A_{p-1}B_p$ (proof with microscopic positivity given in Appendix A.2).

2.3. Nicodin *p*th Squares (Fig. 1d), $a_q=2q-1$, $\vec{v}_i(x, y)$ x, y Odd Integers, without (0, 0):

$$\text{Nicodin: } \vec{v}_i\text{: } (\pm a_q, \pm a_s), (\pm a_s, \pm a_q), (\mp a_q, \pm a_s), (\mp a_s, \pm a_q),$$
$$s=0, 1,..., q, \quad q=1,..., p. \tag{3a$''$}$$

Contrary to the previous models, we have no velocities along the y axis with $x=0$, but still velocities along the bisectors.

Lemma 1$''$ (Physical Nicodin[9] Models). Assuming that the $p-1$th square is physical, we prove that the p one is physical. We still use velocities at the tops of squares and rectangles with three associated densities belonging to a previous physical model. We write only (x, y) for $\vec{v}(x \geqslant -1, y \geqslant -1)$ and use the x, y symmetries for the other densities: As for Class II, in (i) we first add the $(a_p, 1)$ associated density but with $y=1 \neq 0$ and with the x, y exchanges $(1, a_p)$. In (ii) and with the x, y exchanges we add the densities associated to (a_p, a_s), (a_s, a_p), while in (iii) they are the densities with velocities along the bisectors:

$$\begin{aligned} &\text{(i)} \quad (a_{p-1}, 3)+(a_{p-1}, -1)=(a_{p-2}, 1)+(a_p, 1), \\ &\text{(ii)} \quad (a_p, 1)+(a_{p-1}, a_j)=(a_{p-1}, 1)+(a_p, a_j) \quad j=2, 3,..., p-1, \\ &\text{(iii)} \quad (a_{p-1}, a_p)+(a_p, a_{p-1})=(a_{p-1}, a_{p-1})+(a_p, a_p). \end{aligned} \tag{3b''}$$

The pth square is physical, from the $p=2$, $16\vec{v}$ physical,[9] all pth squares are physical.

Isotropic (i) state with equal densities associated to velocities with the same modulus: From $\vec{v}^2(a_q, a_s)=\vec{v}^2(a_s, a_q)=a_q^2+a_s^2$ we have a multiplicity of 8 for $n^{(i)}_{a_q, a_s}$, $s=1,...\, q-1$ and only 4 for $n^{(i)}_{a_q, a_q}$. We write the macroscopic quantities:

$$\begin{aligned} M^{(i)}/4 &= \sum_{q=1}^{p} \left[n^{(i)}_{a_q, a_q} + 2 \sum_{s=1}^{q-1} n^{(i)}_{a_q, a_s} \right], \qquad J^{(i)}=0, \\ E^{(i)}/4 &= \sum_{q} \left[a_q^2 n^{(i)}_{a_q, a_q} + \sum_{s=1}^{q-1} (a_q^2+a_s^2)\, n^{(i)}_{a_q, a_s} \right]. \end{aligned} \tag{4a''}$$

We normalize one density, introduce a parameter m, deduce $n^{(i)}_{a_q, a_s}$, and check (i) in (3b''):

$$\begin{aligned} &n^{(i)}_{1,3}=n^{(i)}_{3,1}=1, \qquad n^{(i)}_{1,1}=m^8, \qquad n^{(i)}_{3,3}=m^{-8}, \\ &n^{(i)}_{1,3}n^{(i)}_{3,1}=n^{(i)}_{1,1}n^{(i)}_{3,3}=1, \qquad n^{(i)}_{a_p, a_q}=m^{10}/m^{a_p^2+a_q^2}, \\ &[\text{i}]\colon n^{(i)}_{a_{p-1}, -1}n^{(i)}_{a_{p-1}, 3}=n^{(i)}_{a_p, 1}n^{(i)}_{a_{p-2}, 1}=m^{20}/m^{2a_p^2-16p+26}. \end{aligned} \tag{4b''}$$

We could also check (ii), (iii). With $b=\log m$, we write the second fictitious (1e) relation:

$$\begin{aligned} 2E^{(i)}/M^{(i)} &= d_p/2b = -\partial_b \Delta_p/\Delta_p, \\ \Delta_p &= \sum_{q=1}^{p} \left[e^{-2ba_q^2} + 2 \sum_{s=1}^{q-1} e^{-b(a_q^2+a_s^2)} \right] = \left[\sum_{q=1}^{p} e^{-ba_q^2} \right]^2, \\ -\partial_b \Delta_p &= \sum_{q=1}^{p} 2 \left[a_q^2 e^{-2ba_q^2} + \sum_{s=1}^{q-1} (a_q^2+a_s^2)\, e^{-b(a_q^2+a_s^2)} \right]. \end{aligned}$$

We define γ_p, δ_p and with some algebra rewrite the energy to mass ratio:

$$\gamma_p := 4 \sum_{q=2}^{p} q(q-1)\, e^{-4bq(q-1)}, \qquad \delta_p := 1 + \sum_{q=2}^{p} e^{-4bq(q-1)},$$

$$d_p/4b = 1 + \gamma_p/\delta_p > 1. \tag{4c''}$$

We note that (4c″) implies $d_p/4b > 1$ or $d_p > 4b$. Numerically we have verified:

$$1 + \gamma_{p_{\max}}/\delta_{p_{\max}} \simeq 1/2b, \qquad d_{p_{\max}} \simeq 2 \qquad \text{for} \quad b \in (10^{-8}, 0.1),$$

but decreases, going to 1 with the same maximal $p_{\max}$. We also find the stable fictitious dimension $d_{p_{\max}}$ greater than 2 for b greater than 0.1. Unfortunately, we do not have a general analytical proof (except for a fixed p number of squares as we shall see in Section 2.5) because when b decreases, the $p_{\max}$ number of squares for stable dimensional values increases and goes to infinity when b goes to 0. For instance, $p_{\max} \simeq 2500$ for $b = 10^{-6}$ and still larger for smaller b. On the contrary, $p_{\max}$ is very small when b is finite.

Lemma 2″. For b fixed, d_p increases with p or $d_p > d_{p-1}$ (proof in Appendix A.3).

2.4. Second DVMs Continuous Relation (1e) with Only Isotropic (i) State

In this subsection, we only compare the (1d) continuous result $d = 2$ with the DVMs (1e) fictitious dimension $d_{p_{\max}}(b)$. No other constraints coming from discrete kinetic theory are included. From p and b fixed we calculate in the energy to mass ratio written previously $d_p/2b$ and (Figs. 2a–c) deduce $d_{p_{\max}}(b)$. We consider the maximal pth square such that the dimension remains stable when p is larger than $p_{\max}$. We recall (Lemmas 2–...) that *for b fixed, the fictitious dimension increases with p, but it can happen that these increases are very small*, not significant and not leading to the true continuous dimension.

These curves illustrate Cercignani's criticism of DVMs (recall that for the planar models $d_{p_{\max}}(b)$ must be 2!!). *If we exclude the intervals* $b \in [0, 1], [0, 0.1]$ *for the Class II and Nicodin models* where the planar dimension is satisfied, *the stable fictitious curves have dimensions different from 2 and not constant.*

For the models with rest-particle (Figs. 2a, b), (Nested, Classes I and II), the limits for the three curves decrease for b larger than 1.5 and are the same for b large, *while for b going to* 0 *they are the $d_H = 0, 1, 2$ of the associated lattices*. We notice: Nested $\subset$ Class I $\subset$ Class II and $d_{\text{Nested}} \leqslant d_{\text{Class I}} \leqslant d_{\text{Class II}}$. When b decreases, $p_{\max}$ increases, and in Fig. 2a we give some values for $b \in (4., 0.1)$. In Fig. 2b, for b smaller than 0.1 and the fictitious dimensions going to the Hausdorff dimensions, we give the increasing $p_{\max}$ values for $b = 10^{-s}$, $s = 1, 2,..., 8$. For Class II $d_{p_{\max}}$ is equivalent to 2, while for Class I, it is equivalent to 1 for s larger than 3. For $b = 10^{-4}$ fixed and Classes I and II, we give fictitious dimensions close to the limits 1 and 2 when p increases up to $p_{\max}$:

$$\left| \begin{array}{cccccc} b = 10^{-4}, p = & 1 & 50 & 150 & 250 & 400 \\ I, d_p = & 2.610^{-4} & 0.2 & 0.85 & \simeq 1 & \simeq 1 \\ II, d_p = & 2.610^{-4} & 0.22 & 1.4 & 1.99 & \simeq 2 \end{array} \right|. \tag{5a1}$$

For the Nested models, the maximal fictitious dimension decreases toward 0. We give small b and $d_{p_{\max}}$ values:

$$\left| \begin{array}{ccccc} \text{Nested}, b = 10^{-s}, s = & 10 & 39 & 91 & 112 \\ (p_{\max}, d_{p_{\max}}) = & (19, 0.087) & (67, 0.022) & (154, 0.009) & (190, 0.0077) \end{array} \right|. \tag{5a2}$$

For the Nicodin model (no rest-particle), except for the interval $b \in (0, 0.1)$ with $d_{p_{\max}} = d_H = 2$, then the stable dimension becomes larger than 2 and increases with b. In Fig. 2c, we give the decreasing $p_{\max}$ for $b \in (0.1, 0.25)$ and here, the $p_{\max}$ and $d_{p_{\max}}$ for smaller and larger b values:

$$\left| \begin{array}{ccccccc} \text{Nicodin: } b = 10^{-s}, s = & 1 & 3 & 4 & 5 & 6 & 7 \\ d \simeq 2, p_{\max} = & 9 & 80 & 246 & 774 & 2443 & 7719 \\ p_{\max} < 6, (b, d_{p_{\max}} > 2) = & (0.25, & 2.004) & (0.5, & 2.28) & (1., & 4.01) \end{array} \right|. \tag{5b}$$

For all pth squares smaller than $p_{\max}$ associated to the presented Class I–II curves, the $d_p(b)$ curves are in the intermediate domains between the Nested curve and either the Class I or Class II curve. For stable dimensions and b varying, we cannot consider a fixed number of p-squares because p increases up to infinity when b decreases to 0, while for b large a small number of pth squares is sufficient. In both Classes I and II for b fixed, we find a similar $p_{\max}$ value for stable dimensions but different from the Nested ones. The Nicodin model without rest-particle is very different from the three models with rest-particle but, like Class II, has a subdomain with stable dimensions equivalent to 2.

A shortcoming of our study is that for models with rest-particle and b larger than 1, or without rest-particle and b larger than 0.1, adding more velocities, the planar dimension 2 cannot be satisfied. This means that we cannot, like Cercignani in the continuous theory, in DVMs with only one asymptotic state, consider only the energy to mass ratio. We must include other DVM constraints, for instance, add another asymptotic (ii) state and check the positivity. *To the (i) isotropic state, we must add the (ii) upstream state and the conservation laws*. As we shall see (Section 3), for the models with rest-particle (Figs. 1a–c), the *main result will be:* $d_{p_{\max}}$ *larger than* $2b$, *leading to the continuous Cercignani requirement* $d_{p_{\max}}$ *equivalent to* 2, *only for Class II*.

2.5. The Four Previous Models with Only a Fixed Number of *p*th-Squares

The standard use of DVMs corresponds to a fixed number of velocities, contrary to above where we increase this number up to stable dimension values. This number $p_{\max}$ increases when b decreases, so that we expect more and more changes when b decreases.

First, we consider the four previous classes of models for a *fixed number p of squares and b varying*. We forget p and rewrite the previous energy to mass ratios as:

$$\Delta_p = \delta_R + \Delta,\ \delta_R = 0 \text{ or } 1/4, \quad \Delta > 0,\ \Delta' < 0,\ \Delta'' > 0, \quad d/2b = \Delta'/(\delta_R + \Delta). \tag{6a}$$

We show, with the Schwarz inequality, that for p fixed, $d/2b$ is decreasing:

$$[\delta_R + \Delta]^2\,(d/2b)' = -\Delta\Delta'' + [\Delta']^2 - \delta_R\Delta'' < 0. \tag{6b}$$

Second, we restrict our study to the *Class II and Nicodin models*, for p and $p_{\max}$ fixed sufficiently large such that the planar dimension 2 exists for b not too small. We let b decrease and rewrite (6a) with derivatives now for both Δ and d:

$$(\delta_R + \Delta)\,d' = -(d+2)\,\Delta' - 2b\Delta''. \tag{6c}$$

When $b = 0$, then Δ, Δ', and Δ'' are finite. In the rhs, the first term is positive and finite while the second is negative, going to 0 when b goes to 0.

For b sufficiently small, d' is positive and the fictitious dimension decreases when b decreases.

For instance, let us start with these two models where $d_p \simeq 2$ respectively for $b = 0.1$, $p = 10$ (Fig. 2c) and $b = 0.5$, $p = 5$ (Fig. 2a). We give the $d_{10}(b)$, $d_5(b)$ values when b decreases:

$$\left| \begin{array}{lllllll} b = 10^{-s} & s = & 1 & 2 & 6 & 10 & 15 \\ \text{Nicodin} & d_{10}(b) = & 2 & 1.84 & 34.10^{-5} & 34.10^{-9} & 34.10^{-14} \\ \text{Class II, } d_5(0.5) \simeq 2, & d_5(b) = & 1.81 & 0.37 & 4.10^{-5} & 4.10^{-9} & 4.10^{-14} \end{array} \right|. \tag{6d}$$

For these $p = 10, 5$ models, the planar dimension $d \simeq 2$ is only in $b \in (0.09, 0.1), (0.5, 1)$.

More generally for each number of squares p fixed, we find for the dimension equivalent to 2, an interval $b \in (b_{\min}, b_{\max})$ and $b_{\min}$ decreases when p increases. However, for $b \in (0, b_{\min})$, the $d_p(b)$ curves decrease when b decreases and go to 0 when b goes to 0.

In conclusion, for *these DVMs with a fixed p number of squares, we can obtain solutions satisfying the second continuous relation with stable dimension equivalent to 2, but only in fixed b intervals*. In the continuous theory, b is inversely proportional to the temperature T or internal energy E_I. At the (i) isotropic state ($J^{(i)} = 0$), then when d is constant, the ratio for the internal energy is the ratio for the corresponding b values. For these DVMs with a finite number of squares, the second continuous relation can be satisfied only in intervals of these macroscopic quantities and the planar 2-dimension solutions are only in fixed b (or T, or E_I) intervals with b not too small.

3. UPSTREAM (ii) STATE AND CONSERVATION LAWS

We add an asymptotic (ii) state and try to see whether positivity restricts the b values.

3.1. $a_{q_j} = 2^{2q_j - 1}$ Nested, $a_{q_j} = q_j$ Classes I and II Models Leading to $d_p \geqslant 2b$

We study two classes of solutions for these models. The nonzero (ii) state densities are either at the x positive or x negative axis. In both cases we find the same result: $d_p \geqslant 2b$, *which for Class II (contrary to Class I and Nested models), excludes the b values with $d_{p_{\max}}$ different from 2.*

We consider traveling waves $\eta = x - \zeta t$, (speed ζ), with isotropic (i) state and densities $n^{(ii)}_{x,y} = n_{x,y} + n^{(i)}_{x,y}$ at the (ii) state associated to $\vec{v}(x, y)$ and only a fixed number of $n^{(ii)}_{-a_{q_j},0} \neq 0$ with velocities $-v_{q_j} = -a_{q_j} < 0$ along the $x<0$-axis. Their number and their location is fixed: $1 \leqslant q_{\min} \leqslant q_j \leqslant q_{\max} \leqslant p$.

In Appendix B.1, for the reader not familiar with DVM conservation laws, we briefly give some explanations. We write in Appendix B.1 and B.2, with microscopic densities, the mass $[M]$, momentum $[J]$, and energy $[E]$ conservation laws that we rewrite here with the two states (i) and (ii):

$$M^{(ii)} = \sum n^{(ii)}_{-a_{q_j},0}, \qquad J^{(ii)} = -\sum v_{q_j} n^{(ii)}_{-a_{q_j},0},$$
$$2E^{(ii)} = \sum v^2_{q_j} n^{(ii)}_{-a_{q_j},0}, \qquad v_{q_j} = a_{q_j} > 0, \tag{7a}$$

$$J^{(ii)} = \zeta(M^{(ii)} - M^{(i)}), \qquad 2E^{(ii)} = \zeta J^{(ii)} + E^{(i)}, \quad 2\zeta(E^{(i)} - E^{(ii)}) = \sum_{q_i} n^{(ii)}_{-a_{q_i},0} v^3_{q_i}. \tag{7b}$$

From the mass conservation: $[M]$, written with microscopic densities first relation in (7b) with macroscopic quantities, we only have $\sum(a_q + \zeta)\, n^{(ii)}_{-a_q,0}$ at the (ii) state giving the macroscopic terms ζM^{ii} and $J^{(ii)}$. At the (i) state, the sum with terms proportional to ζ gives $\zeta M^{(i)}$ and 0 for the other.

Similarly, from the momentum conservation $[J]$, the second relation in (7b), we only have macroscopic quantities. Finally, from the energy conservation $[E]$ we also get macroscopic quantities, except for the terms $a^3_q n^{(ii)}_{-a_q,0}$ in (7b). We define

$$z := M^{(i)}/E^{(i)} > 0, \;\; d_p/2b = 2/z, \bar{n}_j := n^{(ii)}_{-a_{q_j},0}/E^{(i)}, \;\; 1 \leqslant q_{\min} < \ldots q_j \ldots \leqslant q_{\max}, \tag{7c1}$$

giving with (7a,b) for the three conservation laws, microscopic densities $\bar{n}_j$ at the (ii) state:

$$A_j := \bar{n}_j(\zeta + v_{q_j}), \quad 1 = \sum_j A_j/\zeta z = \sum_j v_{q_j} A_j = \sum_j v^2_{q_j} A_j/2\zeta, \quad v_{q_j} = a_{q_j} > 0. \tag{7c2}$$

Lemma 3. From $z>0$, $\bar{n}_j > 0$, and $v_{q_j} > 0$, we deduce $\zeta > 0$, $A_j > 0$, $z \leqslant 2$, and $d_p \geqslant 2b$.

The proof is given in Appendix B.3. We introduce one q_i fixed but arbitrary satisfying $q_{\min} < q_i < q_{\max}$. With the sign of ζ, A_j unknown but $0 < v_{q_j} < v_{q_{j+1}}$, we consider the only three possibilities: (i) $A_{q_{\min}} > 0$, (ii) $A_{q_{\max}} < 0$ and (iii) $A_i < 0$, $A_{i+1} > 0$ for one value of i. *With positivity, only the first case (i) remains*, and we prove Lemma 3.

In the particular case with $q_{\max} = q_{\min} + 1$ *(Class II)*, the proof is still valid with the three cases $A_{q_{\min}} > 0$, $A_{q_{\max}} < 0$, and $A_{q_{\min}} < 0$, $A_{q_{\max}} > 0$.

Let us consider $n^{(ii)}_{a_{q_j},0} \neq 0$ now with velocities along the $x > 0$-axis and number and location fixed: $1 \leqslant q_{\min} \leqslant \dots q_j \dots \leqslant q_{\max} \leqslant p$. The changes are: $A_j = \bar{n}_j(\zeta - v_{q_j})$ and $-v_{q_j}$ in (7c2). From positivity we get $\zeta < 0$, $A_j < 0$, and $z \leqslant 2$, $d_p \geqslant 2b$.

We have chosen some $n^{(ii)}_{(x,0)} \neq 0$ with either $x < 0$ or $x > 0$. However all other densities, not along the x-axis, are 0 or $n^{(ii)} = 0$. *We must verify that no binary collisions with vanishing loss (or gain) term and nonvanishing for gain (or loss) term can exist*. This study is done in Appendix C.

First (Lemma 4) we prove that no collisions with four velocities along the x-axis exist.

Second, for Nested and Class I models, binary collisions with two densities different from 0 along one semi-x-axis and two others equal to 0 either along the bisectors or along the $\vec{y}$-axis cannot exist (Lemma 5).

Third, for the Extended Class I or Restricted Class II models, studied in Section 4, with nonzero (ii) state densities only along one semi-x-axis, we have restrictions.

Finally (Lemma 6), for Class II, we can choose two densities $n^{(ii)}_{p \geqslant 1,0}$ and $n^{(ii)}_{p+S,0}$ with S odd but not for all values. As an illustration, in (C.6) we give some odd values of S which must be excluded and others that are acceptable.

With d_p greater than or equal to $2b$ and comparing with the DVMs curves $d_{p_{\max}}(b)$ of Fig. 2a, we see that *only the Class II, DVMs satisfy the Cercignani continuous constraint* $d \simeq 2$. There remain (Figs. 2a, b), two domains: either b in the interval $(0.1, 1)$ or b smaller than 0.1. For Class II we have almost everywhere $d = d_H \simeq 2$, the second continuous relation, and this answers the Cercignani criticism. For the other models, $d_{p_{\max}}$ decreases, when b decreases, towards $d_H = 1$ for b smaller than 10^{-3} (Class I) or towards $d_H = 0$ (Nested) for b very small and $d \simeq 2$ *not satisfied*.

Finally, we present two other General Results deduced from the mass and momentum conservation laws (7b). We define the mass and energy ratios at the two asymptotic states, recall z, and deduce two relations from (7b): First, from the first two conservation laws we eliminate $J^{(ii)}$ and, second, we rewrite the momentum conservation:

$$\begin{aligned} &z = M^{(i)}/E^{(i)}, \qquad \delta_M = M^{(i)}/M^{(ii)}, \qquad \delta_E = E^{(i)}/E^{(ii)}, \\ &2\delta_M/\delta_E = z\zeta^2(1-\delta_M) + \delta_M, \qquad J^{(ii)}/M^{(ii)} = \zeta(1-\delta_M). \end{aligned} \tag{7e}$$

If now we assume only one or two (ii) states (called infinite or semi-infinite), we can obtain from (7c–e) explicit results for the solutions. This depends also on the number of pth squares at the (i) state (for instance $p = 1$ and only one (i) state leads to the Broadwell[1] solution). However, here we are mainly interested in the models satisfying the Cercignani condition $d_{p_{\max}} \simeq 2$ and so will not consider p fixed. In Appendix B.4, we briefly recall the first continuous relation,[2] deduced from Maxwellians at two asymptotic states with δ_M and δ_T for the mass and temperature (or internal energy) ratios, satisfied in DVMs only by the infinite solutions. In (7f1), the relation is written for $d = 2$, with z, and in (7f2), we add the two (7e) relations. For infinite solutions, the two terms are zero in (7f2):

$$[2\delta_M/\delta_E - z(J^{(ii)}/M^{(ii)})^2][1-3\delta_M] + 2\delta_M[3-\delta_M] = 0, \tag{7f1}$$

$$[1+z\zeta^2(1-d_M)][1-3\delta_M] + 2[3-\delta_M] = 0. \tag{7f2}$$

The main result will be that the parameter b will have explicit values. For brevity, we do not consider the semi-infinite solutions.[2–8] We consider only one (ii) state and Class II, where p is only reduced to p greater than or equal to $p_{\max}$ and $d \simeq 2$.

Infinite Class II Shock Solutions

In (7a–c) with, at the (ii) state, one $\bar{n}$, $A = \bar{n}(q+\zeta)$, $q \geqslant 1$ we get $\bar{n}$, ζ, z, δ_M, and δ_E with (7e). Furthermore we deduce b for $d_{p_{\max}} \simeq 2$:

$$\begin{aligned} &1 = A/z\zeta = qA = q^2A/2\zeta, \qquad \bar{n} = 2/3q^2, \qquad z = 2/q^2, \qquad \zeta = q/2, \\ &z\zeta^2 = 1/2, \qquad J^{(ii)} = qM^{(ii)}, \qquad \delta_M = z/\bar{n} = 3, \\ &\delta_E = 12/(1+\delta_M) = 3, \qquad d_{p_{\max}}/2b \simeq 1/b = 2/z \qquad \text{or} \quad b \simeq q^{-2}. \end{aligned} \tag{7g}$$

The solutions (7g) satisfy (7f2). In fact, for $q = 1$, $d_{p_{\max}} = 1.9959$ we have $b = 0.9979$, while for $q > 1$ the b values are $b = q^{-2}$, leading to an infinite number of fixed b values.

3.2. (ii) State, Conservation Laws for the Nicodin Model without Rest-Particle

We still assume traveling waves $\eta = x - \zeta t$ (speed ζ), isotropic (i) state, and at the (ii) state a fixed number of nonzero densities $n_{x,y} = n_{x,y}^{(ii)} - n_{x,y}^{(i)}$. The conservation laws $[M]$, $[J]$, and $[E]$, in terms of $n_{x,y}$ are written in Appendix B.5. For the mass and the momentum they still depend only on macroscopic quantities, but for the energy conservation we must remake explicit the nonzero densities at the (ii) state. The difficulty is to find solutions with gain and loss collision terms of binary collisions either both vanishing or both nonvanishing. We retain four densities for the fixed number of (ii) state densities: $n^{(ii)}_{a_j,\pm 1} \neq 0$ (only two independent)

$$n^{(ii)}_{a_j,\pm 1}, \quad j = j_1, j_2 = j_1 + 1, \quad a_{j_1} = 2j_1 - 1, \quad a_{j_2} = a_{j_1} + 2, \quad n^{(ii)}_{a_j,1} = n^{(ii)}_{a_j,-1}.$$

We write the mass, momentum, and energy conservation laws deduced from (B.8):

$$M^{(ii)} = 2 \sum_{j_1}^{j_2} n^{(ii)}_{a_j,1}, \qquad J^{(ii)} = 2 \sum a_j n^{(ii)}_{a_j,1},$$
$$E^{(ii)} = \sum_j (a_j^2 + 1)\, n^{(ii)}_{a_j,1}, \qquad a_j = 2j - 1 > 0, \tag{7a'}$$

$$J^{(ii)} = \zeta(M^{(ii)} - M^{(i)}), \qquad 2E^{(ii)} = M^{(ii)} + \zeta J^{(ii)} + E^{(i)},$$
$$\zeta(E^{(ii)} - E^{(i)}) = \sum_j n^{(ii)}_{a_j,1} a_j (a_j^2 + 1). \tag{7b'}$$

We define z... and deduce:

$$z = M^{(i)}/E^{(i)} > 0, \quad d_p/2b = 2/z, \quad \bar{n}_j = n^{(ii)}_{a_j,1}/E^{(i)}, \quad j = j_1, j_2, \quad A_j = \bar{n}_j(\zeta - a_j), \tag{7c'1}$$

$$1 = 2 \sum_{j_1}^{j_2} A_j/\zeta z = -2 \sum a_j A_j = \sum_j (a_j^2 + 1)\, A_j/\zeta, \qquad a_j = 2j - 1 > 0 \quad \text{odd}. \tag{7c'2}$$

Lemma 3′. From $z > 0, \bar{n}_j > 0,\ a_j > 0$, we deduce $\zeta < 0,\ d_p \geqslant 2(a_{j_1}^2 + 1)\, b$. (Appendix B.5). For $j_1 = 1, 2... \to d_p > 4b, 20b,...,$ we recall that $d > 20b$ leads to $d \simeq 2$.

4. EXTENSIONS OF THE MODELS, INTERMEDIATE BETWEEN CLASS I AND CLASS II

We consider two classes of models in Figs. 3 and 4.

First, the r-Extended Class I, pth models, *r finite and fixed* (written for brevity r-Ext$_I$.p), where we add to Class I a number of velocities of the order of const.p but having $\simeq$ const.p^2 velocities less than those of Class II. So $d_H \simeq 1$, with larger d_p than those in Class I; however we cannot satisfy the planar dimension 2.

Second, we consider the Restricted Class II, pth models (written for brevity α-Res$_{II}$.p with α finite), where the number of missing Class II

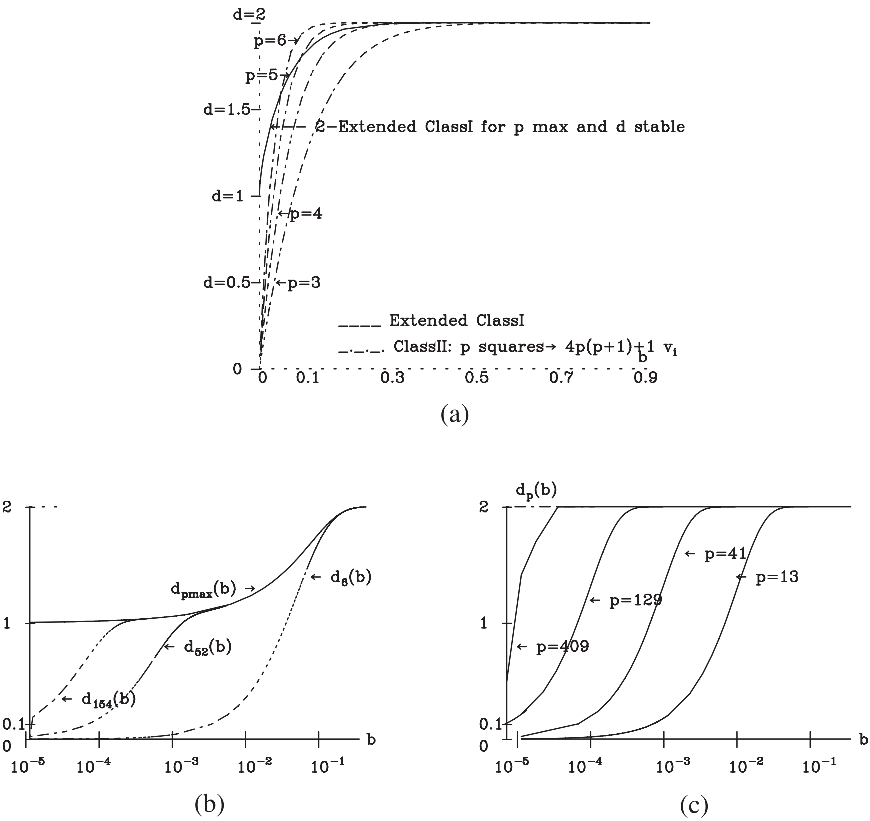

Fig. 4. −·− (a) Class II with a finite number $p = 3, 4, 5, 6$ of squares and −−− 2-Ext.$_I$, Class I, $r = 2$ with p increasing for d stable, (b) −−− 2-Ext.$_I$, Class I, $r = 2$, $b = 10^{-s}$, $s = 1,..., 5$ and −·− $d_p(b)$ for $p = p_{\max} = 6, 52, 154$, (c) $\alpha = 5$ and 5-Rest.$_{II}$ Class II, b$=10^{-r}$, $s = 1,..., 5$.

velocities is finite, $4\alpha(\alpha-1)$ leading to $d_H = 2$ and satisfying the Cercignani planar dimension. We add to the Class I model the integers parallel either to the x or y axes. The main difference between these two classes of models is that the number of these new parallels is either finite or increasing with p.

4.1. r-Extended Class I *p*th Models with $d_{p_{\max}} \simeq 2$ Not Satisfied

Only the $p = 1$ square is common to Classes I and II. To Class I, in addition to the velocities along the coordinates and the bisector axes (x, y, $y = \pm x$ axes), we add the integers parallel to the y-axis wit

$x = \pm 1, \ldots \pm r$ and parallel to the x-axis with $y = \pm 1, \ldots \pm r$ (r finite and fixed). For $r = 1, 2, \ldots$, the $p = 2, 3, \ldots$ squares are common to Class II and the r-$\mathrm{Ext}_I . p$.

Lemma 7a. The r-$\mathrm{Ext}_I . p$th models are physical. We write the new $\vec{v}(x, y)$ and collisions:

$$\begin{aligned} &1\text{-}\mathrm{Ext}_I . p\ \vec{v}\text{: Class I}, (\pm q, \pm 1), (\pm 1, \pm q), q = 2, 3, \ldots, p.(q-1, 1) + (q, 0) \\ &\qquad = (q-1, 0) + (q, 1), \\ &2\text{-}\mathrm{Ext}_I . p\ \vec{v}_i\text{: } 1\text{-Ext}. p \text{ and } (\pm q, \pm 2), (q-1, 2) + (q, 1) \\ &\qquad = (q-1, 1) + (q, 2), q = 3, 4, \ldots, p. \end{aligned} \tag{8a}$$

For $r = 1$, starting first with Class I, we consider successively the new collisions and we can add the densities associated to $(q, 1), q = 2, 3, \ldots p$ (and the symmetric densities with respect to the x, y, $y = \pm x$). Similarly for $r = 2$, starting first with 1-Ext_I we can add successively the densities associated to $(q, 2), q = 3, \ldots p$ (and the symmetry). For $r = 3, 4 \ldots$ Extended... starting with the $r-1$ model, we add successively $(q, r), q = r+1, \ldots p$, and the symmetric densities.

In Figs. 3a, b (another interpretation in Section 4.2), we present the 3-$\mathrm{Ext}_I . 5$,3-$\mathrm{Ext}_I . 11$ models for $r = 3$, $p = 5, 11$. We see that the $q = 1, 2, 3, 4$ squares are common with Class II. The main difference is the number of missing Class II velocities, which is 4(2) for $p = 5$ and 4(8)(7) for $p = 11$. The defect of these models, with r *fixed* is that when p increases, the number of missing Class II velocities increases too much in order to recover $d_{p_{\max}} \simeq 2$. Similarly in Fig. 3c, with $r = 2$, $p = 15$ for the extended model, the number of missing Class II velocities is 4(13)(12). For b not small, we find b intervals with $d \simeq 2$ but $d_{p_{\max}}$ goes to 1 when b goes to 0. This is explained by the Hausdorff dimension $d_H = 1$ with L_r, N_r (size and number of velocities) when p increases up to infinity. Calling N_I and N_{II} the number of velocities for Classes I and II, the differences with N_r are of the orders of either p or p^2:

$$r \geqslant 1 \text{ finite } p \geqslant r+1, \quad N_r = 4(r+1)(2p-r), L_r = 2p, \quad \log N_r / \log L_r \to_{p \to \infty} 1, \tag{8b1}$$

$$\begin{aligned} N_{II} &= 4p(p+1)+1, \qquad N_{II} - N_r = 4(p-r)(p-1-r), \\ N_I &= 8p+1, \qquad N_r - N_I = 4r(2p-r-1). \end{aligned} \tag{8b2}$$

In Fig. 4a we present an example of 2-$\mathrm{Ext}_I . p_{\max}$ $r = 2$ (including Class II, $p = 1, 2, 3$), with $p_{\max}$ in order that $d_{p_{\max}}$ remains stable for higher p values, and we compare with the Class II $d_p(b)$, $p=3, 4, 5, 6$ curves. The $d_3(b)$ is below the Extended curve, while for $p=4, 5, 6$ they are slightly equal or higher in a small b interval, but for b smaller they are both smaller and going to 0 while the Extended goes to 1. Now for $b \in [0.3, 0.95]$, we have $d_{p_{\max}} \simeq 2$, while for larger r we have larger intervals but *always the* $d_H = 1$ *limit when b goes to 0.*

In Fig. 4b, still with 2-Ext_I, $r=2$ for small b values, 10^{-s}, $s=1,\ldots 5$ we present the $d_{p_{\max}}$ curve with $p_{\max} \in (6, 1000)$. We verify that the limit for b very small is the $d_H = 1$ value. In order to have a connection with the finite pth models, we present the $d_p(b)$ curves for $p=6, 52, 154$ and verify that the limits are now 0 when b decreases.

4.2. Restricted Class II, α-Cross pth Models (Fig. 3), with $d_{p_{\max}} \simeq 2$ Satisfied

Instead of r finite in Section 4.1, we define $\alpha = p - r \geqslant 2$ *finite* (r *arbitrary*). p and r increase together and the number of *missing Class II velocities is finite*. The α-cross, $p = \alpha + r$ models have $d_H = 2$ and only a finite number of missing Class II velocities. We rewrite (8b) with $\alpha = p - r \geqslant 2$, and N_r becoming N_α for the α-$\text{Rest}_{II}\,.p$ models:

$$\alpha \geqslant 2 \text{ finite}, \quad p \geqslant \alpha + 1, \quad N_\alpha = 4(p+\alpha)(p-\alpha+1),$$

$$L_\alpha = 2p, \quad \log N_\alpha / \log L_\alpha \to_{p\to\infty} 2, \tag{8c}$$

$$N_{II} - N_\alpha = 4\alpha(\alpha-1) \text{ finite}, \qquad N_\alpha - N_I = 4[p(p-1) + \alpha(\alpha-1)] \to_{p\to\infty} \infty.$$

Comparing with Section 4.1, the important point is that Class II and Restricted models increase similarly when p increases. As an illustration, we write explicitly the missing $\alpha(\alpha-1)$ velocities (see Fig. 3) in the first $x>0$, $y>0$ quadrant. They must be completed with (y, x):

$$\alpha = 2 : (p, p-1), \quad \alpha = 3 : (p, p-1), (p, p-2), (p-1, p-2),$$

$$\alpha = 4 : (p, p-1), (p, p-2), (p, p-3), (p-1, p-2),$$

$$(p-1, p-3), (p-2, p-1),\ldots \tag{8d}$$

$$\alpha : (p, p-1),\ldots (p, p-\alpha+1), (p-1, p-2)\ldots(p-1, p-\alpha+1), (p-2, p-3),\ldots$$

$$(p-2, p-\alpha+1),\ldots (p-\alpha+2, p-\alpha+1)$$

and $(x \to y, y \to x)$ in (8d). We write the velocities (x, y) in the first positive quadrant and they must be completed with symmetries: exchanges $x \to -x$, $y \to -y$ and $(x, y) \to (-x, -y)$:

$$\begin{aligned} &\text{(i)} \quad (x, y): x = 0, 1, \ldots\ p = \alpha + r \to (x, 0), (0, x), (x, x) \\ &\text{(ii)} \quad (x, y): y = 1, 2, \ldots, r, x = y+1, \ldots\ p = \alpha + r \to (x, y) \text{ and } (y, x). \end{aligned} \tag{8e}$$

For the energy to mass ratio at the (i) isotropic state, written in (4b′) for the Class II pth models, we eliminate the contribution coming from the missing velocities:

$$\alpha\text{-cross, } p = \alpha + r \text{ models:} \quad 2E^{(i)}/M^{(i)} = d_p/2b = A_p/(1/4 + B_p),$$

$$B_p := \sum_{q=1}^{p} e^{-bq^2}(1 + e^{-bq^2}) + 2 \sum_{q=1}^{r} \sum_{s=q+1}^{p} e^{-b(q^2+s^2)}, \tag{8f}$$

$$A_p = \sum_{q=1}^{p} e^{-bq^2} q^2 (1 + 2e^{-bq^2}) + 2 \sum_{q=1}^{r} \sum_{s=q+1}^{p} (q^2 + s^2)\, e^{-b(q^2+s^2)}.$$

In Figs. 3a–c, adding the new velocities to the Class I model or equivalently eliminating the Class II missing velocities, we present the α-cross 2, 8, 13, $r = 3, 3, 2$, $p = 5, 11, 15$: 2-Rest$_{II}$.5, 8-Rest$_{II}$.11, 13-Rest$_{II}$.15 models. We still have $4\alpha(\alpha - 1) = 8, 224, 634$ less velocities than for the Class II model, but these numbers do not increase when p increases. With $d_H = 2$, we have numerically verified (α less than 16) for different α-cross models, and $b \in (0, 1)$, p larger than $p_{\max}$ that the *Cercignani condition* $d_{p_{\max}} \simeq 2$ *is satisfied. For b small we find* $p_{\max}$ *independent of* α:

$$\left| \begin{array}{ccccccccc} b & 1 & 0.5 & 0.1 & 10^{-2} & 10^{-3} & 10^{-4} & 10^{-5} & 10^{-6} \\ p_{\max} & \alpha+2 & \alpha+3 & \alpha+8 & 41 & 129 & 409 & 1293 & 4090 \end{array} \right|. \tag{8g}$$

For b larger than 1, as in Classes I and II, we have d_p less than 2, but still with d_p larger than $2b$ (analytical proof not presented), we get $d_{p_{\max}} \simeq 2$.

In Fig. 4c, for b small $\in (10^{-1}, \ldots\ 10^{-5})$, we present for 5-Rest$_{II}$.$p$, $\alpha = 5$ (80 velocities less than Class II), both the $d_{p_{\max}} \simeq 2$ and 4 curves $d_p(b)$ with p fixed: 13, 41, 129, 409. These curves are going to 0 when b decreases. However, we verify for these curves with p fixed that the domains with $d_{p_{\max}} \simeq 2$ increase when p increases.

Lemma 7b. The Restricted Class II, α-Cross Models (Fig. 3d), are physical. For α fixed, we start with the physical Class I model, show that the first $p = \alpha + 1$ is physical and assuming that the $p = \alpha + r - 1$ is physical (r arbitrary), we prove (with three densities of a previous physical model) that the $p = \alpha + r$ model is physical. For brevity (due to the symmetries), we consider only $x \geqslant y > 0$.

(i) $p = \alpha + 1 : (x-1, 1) + (x, 0) = (x-1, 0) + (x, 1)$.

We add successively $(x, y = 1)$ with $x = 2, 3, \ldots \alpha + 1$ and similarly $(x \to y, y \to x)$.

(ii) $p = \alpha + 2 : (2, 1) + (1, 2) = (1, 1) + (2, 2), (x-1, 2) + (x, 1) = (x-1, 1) + (x, 2)$.

First we add $(2, 2)$ and successively $(x, y = 2)$ with $x = 3, \ldots \alpha + 2$ and $(x = 2, y)$ with the changes of x, y.

(iii) $p = \alpha + r : (\alpha + r - 1, 1) + (\alpha + r - 1, -1) = (\alpha + r - 2, 0) + (\alpha + r, 0)$, $(\alpha + r - 1, k) + (\alpha + r, k - 1) = (\alpha + r - 1, k - 1) + (\alpha + r, k)$.

First, we add $(\alpha + r, 0)$. Second, we add successively $(\alpha + r, k)$, $k = 1, 2, \ldots r - 1$. Third, similarly we add $(x, y = r)$, $x = r + 1, r + 2, \ldots r + \alpha - 1$. Finally with the square:

$$(\alpha + r - 1, r) + (\alpha + r, r - 1) = (\alpha + r - 1, r - 1) + (\alpha + r, r),$$

we add $(\alpha + r, r)$, and the α-cross models are physical.

5. REGULAR GRIDS WITH MESH STEPS $\neq 1$

First, we assume that for Class I and II we add, for the $\vec{v}_i$, all half-integer coordinates, but still a uniform distribution, doing the same for the Nicodin model with odd half-integer coordinates. The collisions along rectangles or squares are still valid with $(x, y) \to (x/2, y/2)$:

$$\vec{v} \to \vec{v}/2, \qquad \vec{v}^2 \to \vec{v}^2/4, \qquad E^{(i)} \to E^{(i)}/4,$$
$$b \to 4b, z \to 4z, \qquad d_p = 4b/z \to 4b/z, \qquad d_H \to d_H,$$

and the models are still physical. The relations are invariant with the change of b into $4b$, and this is the only modification in curves d and b of Fig. 2. Our main results $d_{p_{\max}}(b) \simeq 2, \neq 2$ for Class II, Nicodin, and Nested, Class I are not changed. More generally, *for a mesh step h finite*, the only changes are for q into qh, b into b/h^2, z into z/h^2 with, for the $d_p(b)$ curves, the same geometrical structure. For the Class I with the defect for the relation $d_{p_{\max}} = 2$, we do not improve with a more uniform concentration of velocities along the coordinates or bisector axes $(x, y, y = \pm x)$.

6. DISCUSSION

The motivation of this work was to answer Cercignani's criticism of DVMs. In the continuous theory, the energy to mass ratio, written with gaussians $e^{-b\vec{v}^2}$, is $d/2b$, where the dimension is $d=2$ for planar models and only $d_p/2b$ in DVMs.

First, in Sections 2 and 3 (Fig. 1), we have considered, for the associated DVMs ratios, four different classes of elementary pth squares models, with the possibility of p going to infinity for the associated lattices:

(i) models with a rest-particle and Hausdorff dimensions $d_H = 0, 1, 2$: Nested with a nonuniform distribution along the coordinates and bisector axes (x, y, $y = \pm x$), Class I with a uniform distribution along these axe, and Class II, filling all integer coordinates when p is going to infinity;

(ii) for models without rest-particle, the Nicodin model with $d_H = 2$ fills odd-integer coordinates.

Second, in Section 4, we have studied two new classes of models which are intermediate between Class I and II:

(i) Extended Class I with $d_H = 1$ (an infinite number of velocities less than those of Class II).

(ii) Restricted Class II or α-Cross (α fixed) models (Fig. 3), with $d_H = 2$ and only a finite number of velocities less than those of Class II.

For these DVMs, we write for the pth square model, the energy to mass ratio at an equilibrium (i) isotropic state (with sums of gaussians $\sum e^{-b\vec{v}_i^{\,2}}$) giving $d_p/2b$ a *fictitious dimension* d_p. For b fixed, and p increasing, the d_p are going to a stable $d_{p_{\max}}$ dimension, and these limits are fixed.

Then, for the models in Fig. 1, *we compare these limits and the true $d = 2$ dimension*. We find that this is possible only in a fixed b interval $b \in (0, b_{\max})$ and only for the Class II and Nicodin models with $d_H = 2$. For the other Nested and Class I models, we find $d_{p_{\max}}$ less than 2 and the limits, when b is going to 0, are the $d_H = 0, 1$ dimensions of the associated lattices.

However, for Class II and Nicodin models, when b is larger than $b_{\max}$, these $d_{p_{\max}}$ limits are different from the planar $d = 2$ dimension, even when the number of velocities is increasing or the mesh steps are smaller.

So Cercignani's criticism of DVMs is still partly valid. This means that it is not sufficient to discretize, even with physical models, and that some important properties of the kinetic theory are still missing.

For traveling DVM waves, we add to the above isotropic (i) downstream state, *an upstream (ii) state and with conservation laws and positivity,* we study the new DVM constraints. We find mainly that the intervals with b larger than $b_{\max}$ must be excluded. It follows for Class II and Nicodin models (Figs. 1c–d) (contrary to Class I and Nested models) that d equivalent to 2 is satisfied. Similarly, for the Restricted Class II with $d_H = 2$ (Fig. 3) (contrary to the Extended Class I with $d_H = 1$), we find d equivalent to 2 satisfied.

In conclusion, in order to satisfy Cercignani's criticism of DVMs for the continuous energy to mass ratio, we must consider *only models with* $d_H = 2$ *and add the constraints of the Rankine–Hugoniot DVMs conservation laws*. We notice that the Class II and Nicodin models (Figs. 1c, d), have a uniform distribution of the velocities in the plane. But this is not necessary, because this property does not hold for the Restricted Class II (Fig. 3), satisfying also the continuous dimension condition.

We keep in Fig. 3, for these Restricted α models, the velocities along the bisectors $y = \pm x$. However, in the small squares of length α, including the missing Class II velocities, we could also eliminate these velocities along the bisectors, giving now $4\alpha^2$ missing velocities, still a finite number. These new models (for the sake of brevity, the results are not presented, but it is sufficient to replace $\sum_{q=1}^{p}$ by $\sum_{q=1}^{r}$ in Eq. (8f)) have $d_H = 2$ and satisfy the condition d equivalent to 2.

The results have been obtained mainly with mathematical tools (lemmas) and numerical calculations (for instance when b decreases, then $p_{\max}$ increases for $d_{p_{\max}}$). What can we understand physically? These p increases when b decreases mean that the results become close to the associated lattices with Hausdorff dimensions 0, 1, 2 (Fig. 2b). Consequently, we understand (Figs. 4a–c) why the Extended Class I and the Restricted Class II give similar results to Classes I and II. But when b is large we find with only the mass to energy ratios (contrary to the continuous theory) either $d_{p_{\max}}$ decreases with the same $d_{p_{\max}}$ values (Fig. 2a) for our models with rest-particle (different Hausdorff dimensions), or increases as the Nicodin model without rest-particle. The explanation is that for b large the $p_{\max}$ (Fig. 2b), are very small and, consequently, the links with the associated lattices are missing. Now these defects disappear when we require the constraints of the conservation laws, adding an upstream (ii) state. However, in our Lemmas 3 and 3′ the new constraints are mainly due to positivity and were not necessary for b small, and so are not easy to understand physically.

The standard use of DVMs being a finite number of velocities or p fixed, we notice that for b fixed, the $d_p(b)$ curves increase with p increasing. For b not small the $d_{p_{\max}}(b)$ limits (with increasing squares) require only a finite number of squares. On the contrary, when b decreases we need more and more squares, so that p must go to infinity when b is going to 0 and we must exclude the standard DVMs in the *discussion* with b going to 0. *In all cases we have found that the dimensional limit values are the same as the Hausdorff dimensions*. We recall that in the continuous theory, b is inversely proportional to T (or E_I). So we can understand that for b going to 0, we recover the classical results associated to the lattice dimension. *For a fixed p number of squares and velocities, we must consider b finite, in fact b (or* T, E_I*) belonging to finite intervals*. If we restrict ourselves to the dimensional physical constraint $d \simeq 2$, there remains only the *Class II, Nicodin, and Restricted Class II models in finite b intervals but larger and larger when p increases*.

Here, we have mainly considered the second continuous relation in order to check the geometrical structures at the (i) state leading to d equivalent to 2 when we add some (ii) state restrictions. For a study of particular classes of (ii) state densities we have mainly studied the infinite solutions with the first continuous relation satisfied. For solutions with two densities different from zero along the semi x-axes, we only give conditions in order to avoid collisions with loss and gain terms that are zero and different from zero. For a complete (ii) state study (for brevity not done here) we will retain only the models giving d equivalent to 2 at the (i) state.

In the present paper, we retain only the Maxwellian continuous energy to mass ratio relations and the Rankine–Hugoniot DVM relations for *physical models* (here only three invariants: mass, momentum, and energy, no more, no less). For the study we choose selected distributions of velocities in the plane. Here, in all models, with the simple tool[6–8] of collisions in rectangles or squares, we prove for all sets of pth squares that they are physical, and this is not always obvious, in particular when p goes to infinity.

With the exception of the Nested[2] models, we have a regular grid for our models with only a fixed mesh step 1. We have seen that with other uniform distributions of the velocities with mesh steps different from 1 (but finite), with scalings of the b and z parameters, we do not change the geometrical structure of the $d(b)$ curves or the solutions of our models.

It will be useful to generalize the present results in the $d = 3$ dimension; we think that we will encounter analogies with lattices in the 0, 1, 2, and 3 dimensions and that the best choice will be the models with $d_H = 3$.

For half-space DVM models[3, 4, 9–11], we have no velocities at the interface with $x = 0$; here this means no velocities, like the Nicodin model, along the y-axis. The nontrivial proof is to determine the selected velocities such that the partial pth models are physical. This was done for the Class II[11] models. Concerning the present study of Cercignani's criticism of DVMs, the results certainly still hold because the Hausdorff dimension is still $d_H = 2$.

APPENDIX A: (FOR b FIXED, d_p INCREASES WITH p)

A.1. Nested and Class I Models: $d_p > d_{p-1}$ or with A_p, B_p Written in (4c1–2):

$A_p(1/4+B_{p-1}) > A_{p-1}(1/4+B_p)$? or $Z=(A_pB_{p-1}-A_{p-1}B_p)\,e^{ba_q^2}>0$ with:

$$Z > \sum_{q=1}^{q=p-1} e^{-ba_q^2}[e^{-ba_q^2}(a_p^2-2a_q^2)+e^{-ba_p^2}(2a_p^2-a_q^2)$$
$$+(1+2e^{-b(a_p^2+a_q^2)})(a_p^2-a_q^2)]>0.$$

Except for the first term, the others are positive, but the Nested first term $a_p^2-2a_q^2$ with $a_q=2^{q-1}$ is positive for $q\leqslant p-1$. There remains Class I: $a_q=q$ and p^2-2q^2, which can be negative. We keep the first term and a lower bound for the last one >0:

$$X := 2q^2-p^2, \quad Ze^{bp^2} > \sum_{q=1}^{q=p-1} [(p^2-q^2)\,e^{b(p^2-q^2)}-Xe^{-bX}]>0? \quad \text{(A.1)}$$

We change q in $p-q$ in the term with X: giving $X := 2q^2+p^2-4pq$ and still $1\leqslant q\leqslant p-1$.

(i) If $q/p>1-\sqrt{2}\,/2$ or $X<0$ or $Z>0$.

(ii) If $q/p<1-\sqrt{2}\,/2$, $X>0$, $p^2-q^2-X=q(4p-3q)>0$, $Z>0$, $d_p>d_{p-1}$.

A.2. Class II model: $d_p > d_{p-1}$ or $Z=A_pB_{p-1}-A_{p-1}B_p>0$

For simplicity in the analytical proof, we define α_p, β_p, rewrite in (4b′), both A_p, B_p, the energy to mass ratio, and Z with these new parameters and we show that $Z>0$:

$$\alpha_p := \sum_1^p q^2e^{-bq^2}, \ \beta_p := \sum_1^p e^{-bq^2}, \ A_p=\alpha_p(1+2\beta_p)>A_{p-1}, \ B_p=\beta_p(1+\beta_p),$$

$$d_p/2b = A_p/(1/4+B_p)=\alpha_p(1+2\beta_p)/[1/4+\beta_p(1+\beta_p)], \quad \text{(A.2)}$$

$$Z = \alpha_p(1+2\beta_p)\,\beta_{p-1}(1+\beta_{p-1})-\alpha_{p-1}(1+2\beta_{p-1})\,\beta_p(1+\beta_p)=\sum_{i=1}^3 Z_i>0.$$

We prove both $Z_1 > 0$ and $Z_2 + Z_3 > 0$:

$$Z_1 = (1+2\beta_p\beta_{p-1})(\alpha_p\beta_{p-1} - \alpha_{p-1}\beta_p)$$

$$= (1+2\beta_p\beta_{p-1}) \sum_1^p e^{-b(p^2+q^2)}(p^2-q^2) > 0,$$

$$Z_2 = 2\beta_p\beta_{p-1}(\alpha_p - \alpha_{p-1}) = 2p^2 e^{-bp^2}\beta_{p-1}(\beta_{p-1} + e^{-bp^2}), \tag{A.3}$$

$$Z_3 = \alpha_p\beta_{p-1}^2 - \alpha_{p-1}\beta_p^2 = e^{-bp^2}[p^2\beta_{p-1}^2 - \alpha_{p-1}(e^{-bp^2} + 2\beta_{p-1})],$$

$$e^{bp^2}(Z_2 + Z_3) = \sum e^{-bq^2}[(3p^2-2q^2)\,\beta_{p-1} - q^2 e^{-bp^2}] > 0 \quad \text{or } d_p > d_{p-1}.$$

A.3. Nicodin Model: For b fixed, $d_p > d_{p-1}$ or with (4c″) $\gamma_p \delta_{p-1} > \gamma_{p-1}\delta_p$ or

$$p(p-1)\,\delta_{p-1} - \gamma_{p-1}/4 > 0$$

$$\text{or} \qquad p(p-1) + \sum_{q=2}^{p-1} (p-q)(p+q-1)\, e^{-4bq(q-1)} > 0. \tag{A.4}$$

APPENDIX B: DVM CONSERVATION LAWS

B.1. DVMs for Nested-Class I models[2-8] with (4a)–(7a) for $M^{(k)}$, $J^{(k)}$, $E^{(k)}$

We consider similarity solutions, with $\eta = x - \zeta t$ along the x-axis and rest-particle r. Two densities with $\vec{v}(x, \pm y)$ are equal. For a qth square, we add 5 independent densities: $n_{\pm x,0}$, $n_{0,x}$, $n_{\pm x,x}$, $x = a_q$. Here we write the mass, momentum, and energy conservation laws $[M]$, $[J]$, $[2E]$: and in (7a, b) deduce the two asymptotic (i), (ii) states relations from $n_{x,y}^{(ii)} = n_{x,y}^{(i)} + n_{x,y}$. We briefly give some explanations. For the mass sum of $n_{x,y}$ terms, from $\partial_t + x\partial_x$ applied to $n_{x,y}(\eta = x - \zeta t)$, we deduce $(x-\zeta)\, n_{x,y}$ terms. So for the rest-particle and $n_{0,y}$ along the y-axis we only get $-\zeta n_{0,y}$, while those with opposite x values give $(\pm x - \zeta)\, n_{\pm x,y}$. For $[J]$, we multiply these terms by the x projections along the x-axis, while for $2[E]$, we multiply by $x^2 + y^2$. We define $[M]^{\pm}$,... associated to the densities $n_{\pm x,y}$:

$$
\begin{aligned}
[M]^{\pm} &= \sum_{1}^{p} (\pm a_q - \zeta)(n_{a_{\pm q},0} + 2n_{a_{\pm q},a_q}),\\
0 = [M] &= -\zeta(r + 2\sum_q n_{0,a_q}) + [M]^{+} + [M]^{-},\\
[J]^{\pm} &= \sum_{1}^{p} \pm a_q(\pm a_q - \zeta)(n_{a_{\pm q},0} + 2n_{a_{\pm q},a_q}),\\
0 = [J] &= [J]^{+} + [J]^{-},\\
[2E]^{\pm} &= \sum_{1}^{p} a_q^2(\pm a_q - \zeta)(n_{a_{\pm q},0} + 4n_{a_{\pm q},a_q}),\\
0 = [E] &= \sum_{1}^{p} a_q^2 n_{0,a_q} + [E]^{+} + [E]^{-}.
\end{aligned}
\tag{B.1}
$$

B.2. DVMs for Class II Models[8] $a_q = q$, with (4a′)–(7a) for $M^{(k)}$, $J^{(k)}$, $E^{(k)}$

$(4q+1)$ independent $n_{x,y}$ for Class II, adding $n_{\pm q,s}, n_{\pm s,q}$, $s = 1, 2,\ldots$ $q-1$,

$$
\begin{aligned}
[M]^{\pm} &= 2\sum_q \sum_{s=1}^{q-1} (\pm q - \zeta)\, n_{\pm q,s} + (\pm s - \zeta)\, n_{\pm s,q},\\
[M] &= \cdots + [M]^{+} + [M]^{-},\\
[J]^{\pm} &= 2\sum_q \sum_s \pm q(\pm q - \zeta)\, n_{\pm q,s} \pm s(\pm s - \zeta)\, n_{\pm s,q},\\
[J] &= \cdots + [J]^{+} + [J]^{-},\\
[E]^{\pm} &= \sum_q \sum_s (q^2 + s^2)[(\pm q - \zeta)\, n_{\pm q,s} + (\pm s - \zeta)\, n_{\pm s,q}],\\
[E] &= \cdots + [E]^{+} + [E]^{-}.
\end{aligned}
\tag{B.2}
$$

B.3. Proof of $d_p \geqslant 2b$ for the Nested, Classes I and II Models

Lemma 3. From $z > 0$, $\bar{n}_j > 0$, $v_{q_j} > 0$, we deduce $\zeta > 0$, $A_j > 0$, $z \leqslant 2$ and $d_p \geqslant 2b$:

First, with $A_j = \bar{n}_j(\zeta + v_j)$ we rewrite the three relations in (7c), introducing an arbitrary q_i:

$$\begin{aligned} &(a) \quad \sum_{q_j=q_{\min}}^{q_i} A_j + \sum_{q_j=q_i+1}^{q_{\max}} A_j = \zeta z, \\ &(b) \quad \sum_{q_j=q_{\min}}^{q_i} A_j v_{q_j} + \sum_{q_j=q_i+1}^{q_{\max}} A_j v_{q_j} = 1, \\ &(c) \quad \sum_{q_j=q_{\min}}^{q_i} A_j v_{q_j}^2 + \sum_{q_j=q_i+1}^{q_{\max}} A_j v_{q_j}^2 = 2\zeta, \quad v_{q_j} > 0, \quad q_{\min} < q_i < q_{\max}. \end{aligned} \tag{B.3}$$

With $\zeta + v_j < \zeta + v_{j+1}$, we consider three cases: (i) $A_{q_{\min}} > 0$; (ii) $A_{q_{\max}} < 0$; (iii) $\zeta + v_i < 0, \zeta + v_{i+1} > 0$ or $A_i < 0, A_{i+1} > 0$. Positivity gives only case (i) possible:

(i) If $v_{q_{\min}} + \zeta > 0$, we get $A_{q_{\min}} > 0$, all $A_j > 0$, and from (a): $z\zeta > 0$ or $\zeta > 0$.

(ii) If $v_{q_{\max}} + \zeta < 0$, we get $\zeta < 0$ and with $v_{q_j} < v_{q_{\max}}$, all $A_j < 0$. The lhs of (b) is negative while the rhs is positive. *This case is not possible.*

(iii) There remains for one v_{q_i}: $A_i < 0$ while, due to $v_{i+1} > v_i$, $A_{i+1} > 0$. Then due to $v_j > v_{i+1}$ for $j > i+1$, and $v_j < v_i$ for $j < i$, we get two sets of $A_j < 0$ or $A_j > 0$:

$$\zeta < 0, \ A_{q_{\min}} < 0, \ A_j < 0 \quad \text{for } j \leqslant i, \ A_{i+1} > 0, \ A_j > 0 \quad \text{for } i+1 \leqslant j \leqslant q_{\max}.$$

We multiply (b) by v_{q_i+1} and subtract (c), giving:

$$\sum_{q_1}^{q_i} v_{q_j}(v_{q_{i+1}} - v_{q_j}) A_j + \sum_{q_{i+1}}^{q_{\max}} v_{q_j}(v_{q_{i+1}} - v_{q_j}) A_j = v_{q_{i+1}} - 2\zeta, \quad \zeta < 0.$$

The rhs is positive while the lhs is negative with the two terms being products of positive and negative factors. *So this case is not possible.*

There remains only the first case of (i) with all $A_j > 0$ and $\zeta > 0$ for (a), (b), and (c). From both the ratio (a)/(b), $v_{q_{\min}} \leqslant v_{q_j} \leqslant v_{q_{\max}}$ and the link between z and d_p we get:

$$\begin{aligned} \sum A_j / \sum A_j v_{q_j}^2 = z/2 \qquad &\text{or} \quad v_{q_{\max}}^{-2} \leqslant z/2 \leqslant v_{q_{\min}}^{-2} \leqslant 1 \qquad \text{or} \quad z \leqslant 2, \\ &\text{or} \quad d_p = 4b/z \geqslant 2b. \end{aligned} \tag{B.4}$$

B.4. Continuous Theory Mass Ratio Relation[2] with d-Dimensional Maxwellian

$$M(2\pi T)^{-d/2}\exp-\left[(v_1-V)^2+\sum_2^d v_i^2\right]\Big/2T \quad \text{with } M^{(k)}, T^{(k)}, V^{(k)}, \quad k=i, ii$$

for the masses, temperatures, and velocities at two asymptotic (i), (ii) states. We write δ_M and δ_T for the mass and temperature (or internal energy) ratios. From the conservation laws we get:

$$MV=\text{const}, \quad M(V^2+T)=\text{const}, \quad MV(V^2+(d+2)\,T)=\text{const}. \tag{B.5}$$

We eliminate $V^{(ii)}$ in the first two relations and $V^{(i)}$ in the last one:

$$\begin{gathered} V^{(ii)}=V^{(i)}M^{(i)}/M^{(ii)}\to (V^{(i)})^2\,(\delta_M-1)=T^{(i)}-T^{(ii)}/\delta_M, \\ 1<\delta_M=(d+1+1/\delta_M\delta_T)/(1+(d+1)/\delta_M\delta_T)\leqslant d+1,\ \delta_M=d+1 \quad \text{for } \delta_T=0. \end{gathered} \tag{B.6}$$

In DVMs, except for infinite shocks, this relation is not satisfied and gives a constraint.[2] We write (B.6) for the isotropic (i) state and for δ_T, replace the ratio with internal energy: $dE_I=2E/M-(J/M)^2$, where $J^{(i)}=0$. We write $\delta_E=E^{(i)}/E^{(ii)}$ for the energy ratio:

$$[2\delta_M/\delta_E-(J^{(ii)}/M^{(ii)})^2\,M^{(i)}/E^{(i)}][1-(d+1)\,\delta_M]+2\delta_M[d+1-\delta_M]=0. \tag{B.7}$$

(B.7) with $d=2$ and $z=M^{(i)}/E^{(i)}$ gives the relation (7f1).

B.5. DVMs for the Nicodin Model: (4a″)–(7a′) for $M^{(k)}$, $J^{(k)}$, $E^{(k)}$

Still Similarity Solutions

$\eta=x-\zeta t$ along the x-axis but without rest-particle and $\vec{v}$ along the x-axis. Two densities with $\vec{v}(x,\pm y)$ are equal. We write the energy conservation laws for densities with two asymptotic (i) isotropic, (ii) states, $n_{x,y}^{(ii)}= n_{x,y}^{(i)}+n_{x,y}$:

$$\begin{aligned} [M]^{\pm}/2=&\sum_{q=1}^{p}(\pm a_q-\zeta)\,n_{\pm a_q,a_q}+\sum_q\sum_{s=1}^{q-1}(\pm a_q-\zeta)\,n_{\pm a_q,a_s} \\ &+(\pm a_s-\zeta)\,n_{\pm a_s,a_q}, \\ [J]^{\pm}/2=&\sum_1^p \pm a_q(\pm a_q-\zeta)\,n_{\pm a_q,a_q}+\sum_1^p\sum_{s=1}^{q-1}\pm a_q(\pm a_q-\zeta)\,n_{\pm a_q,a_s} \\ &\pm a_s(\pm a_s-\zeta)\,n_{\pm a_s,a_q}, \end{aligned}$$

$$[E]^{\pm}=\sum_{1}^{p} 2a_q^2 n_{\pm a_q, a_q}(\pm a_q-\zeta)$$

$$+\sum_q \sum_s (a_q^2+a_s^2)[(\pm a_q-\zeta)\, n_{\pm a_q, a_s}+(\pm a_s-\zeta)\, n_{\pm a_s, a_q}],$$

$$[M]=[M]^{+}+[M]^{-},\quad [J]=[J]^{+}+[J]^{-},\ [E]=[E]^{+}+[E]^{-}, \qquad \text{(B.8)}$$

Lemma 3′. $d_p=4b/z \geqslant 2b(a_{j_1}^2+1)$. We rewrite (7c′) with $A_j=\bar{n}_j(\zeta-a_j)$, $\bar{n}_j>0$:

$$1=2\sum_{j_1}^{j_2} A_j/\zeta z=-2\sum a_j A_j=\sum_j (a_j^2+1)\, A_j/\zeta,\ a_j=2j-1>0,\ z>0,\ a_j>0. \qquad \text{(B.9)}$$

With $j=2j-1, =j_1, j_1+1$ and the unknown ζ sign, we consider the only four possibilities:

(i) $A_{j_i}>0$, $i=1,2$, which is not compatible with the second relation in (B.9);

(ii) $A_{j_i}<0$ giving $\zeta<0$ with the last relation in (B.6);

(iii) $A_{j_1}>0$, $A_{j_2}<0$ with (B.6) giving $a_{j_1}<\zeta<a_{j_2}$ and $z>0$;

(iv) $A_{j_1}<0$, $A_{j_2}>0$ giving with (B.9) $a_{j_2}=a_{j_1}+2<\zeta<a_{j_1}$, which is incompatible.

There remains only (ii), (iii) with $A_{j_2}<0$ and $\zeta \lesseqgtr 0$. From the last two relations in (B.9), we deduce A_{j_i} and define $X:=2(a_{j_1}^2+2a_{j_1}-1)>0$:

$$A_{j_2}=(a_{j_1}^2+1+2\zeta a_{j_1})/2X, \qquad A_{j_1}=-[a_{j_1}^2+4a_{j_1}+5+2\zeta(a_{j_1}+2)]/X. \qquad \text{(B.10)}$$

If ζ is positive, case (iii), we deduce $A_{j_2}>0$, and this is not possible.

There remains (ii), with $A_{j_i}<0$, $\zeta<0$ leading to two conditions: from (B.10):

$$A_{j_2}<0 \quad \text{or} \quad -\zeta>\alpha:=(a_{j_1}^2+1)/2a_{j_1},$$

$$A_{j_1}<0 \quad \text{or} \quad -\zeta<\beta=(a_{j_1}^2+4a_{j_1}+5)/2(a_{j_1}+2),$$

while the remaining condition $\beta>\alpha$ is similar to $X>0$. We eliminate ζ in (B.9):

$$\sum A_{j_i}/\sum A_{j_i}(a_{j_i}^2+1)=z/2 \text{ giving } (a_{j_2}^2+1)^{-1}\leqslant z/2 \leqslant (a_{j_2}^2+1)^{-1},$$

$$2b(a_{j_2}^2+1)\geqslant d_p=4b/z\geqslant 2b(a_{j_1}^2+1). \qquad \text{(B.11)}$$

APPENDIX C: NESTED, CLASS I-II MODELS WITH $n^{(ii)}_{x,0} \neq 0$ FOR ONLY $x > 0$ OR ONLY < 0

Lemma 4. For the Nested-Class I-II (ii) state, no binary collisions along the x-axis exist. From the energy and momentum conservations, collisions with $(p, 0)$, $(q, 0)$, $(r, 0)$, and $(s, 0)$ (four different velocities) are not possible.

$$p+q=r+s, \qquad p^2+q^2=r^2+s^2, \qquad pq=rs \rightarrow p-r=(p-r)(s/p)^2$$
$$\text{or} \quad p=r \qquad \text{or} \quad =s. \tag{C.1}$$

Lemma 5. For Nested, Class I models, collisions between $\vec{v}_i$ along one semi-x-axis $(y=0)$ and (i) either bisectors $y=\pm x$ $((x, \pm x))$ or (ii) the $\vec{y}$-axis are not possible:

$$\begin{aligned}
&p>0, q>0\\
&\text{(i)} \quad (p,0)+(q,0)=(r,s)+(t,u)\\
&\quad \text{or} \quad s=-u=r=t \qquad \text{or} \quad (p,0)+(q,0)\\
&\quad =(s,s)+(s,-s) \rightarrow p+q=2s,\ p^2+q^2=4s^2 \rightarrow pq=0\\
&\quad \text{or} \quad p=0 \qquad \text{or} \quad q=0.\\
&\text{(ii)} \quad (p,0)+(q,0)=(0,r)+(0,s) \rightarrow p+q=0 \rightarrow\\
&\quad \text{either } p \qquad \text{or} \quad q<0.
\end{aligned} \tag{C.2}$$

However, for r-Extended Class I, with densities only $n^{(ii)}_{p,0} \neq 0$, $n^{(ii)}_{p+s,0} \neq 0$, s integer, along one semi-x axis and 0 in the plane, we have restrictions. We begin with $r=1$:

$$(p,0)+(p+s,0)=(p+\lambda, r)+(p+s-\lambda, -r) \qquad \lambda(s-\lambda)=r^2.$$

For $r=1$ and $\lambda=1, s=2$, we see that such collisions with loss or gain terms different from or equal to 0 are possible. Similarly for $r=2$ and either $\lambda=1, s=5$ or $\lambda=2, s=4$....

Lemma 6. For Class II with only $n^{(ii)}_{p,0} \neq 0, n^{(ii)}_{p+1,0} \neq 0$, p integer, no collisions with only $n^{(ii)} \neq 0$ in loss (or gain) term exist. A similar property also holds for $n^{(ii)}_{p,0} \neq 0, n^{(ii)}_{p+S,0} \neq 0$, *but with only particular odd values of* S.

1. For $(p, 0), (p+2s)$, collisions exist with momentum and energy relations satisfied:

$$(p,0)+(p+2s,0)=(p+s,s)+(p+s,-s) \ s=1,2,\ldots \text{ integer.} \tag{C.3}$$

For these collisions if both $(p, 0)$ and $(p+2s, 0)$ have $n^{(ii)} \neq 0$, we have gain (or loss) terms $\neq 0$, but loss (or gain) term $=0$ and similarly with $(p+1, 0)$ and $(p+1+2s, 0)$.

So we can choose $(p, 0), (p+1, 0)$, $p=1, 2,...$ with $n^{(ii)} \neq 0$, but we must exclude $(p, 0), (p+S, 0)$ with S even. It remains possible to have collisions but with loss and gain terms $=0$:

$$(p, 0)+(p+1, r)=(p, r)+(p+1, 0), \qquad r=\pm 1, \pm 2, \pm 3. \tag{C.4}$$

2. To $(p, 0) \neq 0$, *can we add* $(p+S, 0) \neq 0$, S *odd without collisions with vanishing loss or gain terms?* We write possible collisions with other densities equal to 0 in the plane, giving an energy condition:

$$(p, 0)+(p+S, 0)=(p+\lambda, r)+(p+S-\lambda, -r) \qquad \lambda(S-\lambda)=r^2, \tag{C.5}$$

with S odd and λ and r arbitrary integers. With for the first term, the lhs different from zero and the rhs equal to zero, we must exclude the $(p+S, 0)$ densities satisfying (C.5), except for values where the energy condition (second term) cannot be satisfied.

For the energy condition with λ, $S-\lambda$ *either odd, even or even, odd, but* $\lambda(S-\lambda)$ *even, we deduce* r *even*. It is sufficient to start with $r=2, 4, 6, 8...$ and to verify whether λ and S odd are possible. As an illustration, we give for $1<S$ odd <30 the possible S, λ, and r values satisfying (C.5):

$$\begin{vmatrix} S & 5 & 13 & 15 & 17 & 25 & 25 & 29 \\ \lambda & 1 & 9 & 3 & 1 & 5 & 9 & 4 \\ r & 2 & 6 & 6 & 4 & 10 & 12 & 10 \end{vmatrix}. \tag{C.6}$$

Consequently, we must exclude these (C.6) $(p, 0)$ and $(p+S, 0)$. However, we can consider, for instance, $(p, 0)$ and $(p+S, 0)$ with $S=1, 3, 7, 9, 11, 21, 23, 27<30$.

ACKNOWLEDGMENTS

I thank C. Cercignani for his interest in the present work. I thank R. Balian, who suggested comparing the limits of DVMs infinitely many velocities with the Hausdorff dimension.

REFERENCES

1. J. E. Broadwell, Shock structure in a simple discrete velocity gas, *Phys. Fluids* 7:1243 (1963).

2. H. Cornille, Hexagonal DVMs with and without rest-particles, *J. Stat. Phys.* **81**:335–346 (1995); Nested-hexagons and squares DVMs with arbitrary number of velocities satisfying a continuous theory relation; DVMs with arbitrary number of velocities and two continuous theory relations, VIIIth and IXth Int. Conf. Waves Cont. Media, *Circolo Mat. Palermo*, **II 45**:157 (1996); **II 57**:177–182 (1998); Nested-squares discrete Boltzmann models with arbitrary number of velocities satisfying a continuous theory relation, *TTSP* **26**:359 (1997).
3. Some review papers about DVMs: R. Gatignol, Kinetic theory boundary conditions for discrete velocity gas, *Phys. Fluids* **20**:2022–2030 (1977); H. Cabannes, *Discrete Boltzmann Equation*, Lect. Notes, Vol. 00 (University of California, Berkeley, 1980); D. d'Humieres, P. Lallemand, and U. Frisch, Lattice gas models for 3D hydrodynamics, *Europh. Lett.* **2**:291–297 (1986); T. Platkowski and R. Illner, *SIAM Rev.* **30**:213 (1988); N. Bellomo and T. Gustafson, The discrete Boltzmann equation: Aspects of the initial-boundary value Pb, *Rev. Math. Phys.* **3**:137 (1991); R. Monaco and L. Preziosi, *Fluid Dynamic Applications of the Discrete Boltzmann Equation* (World Scientific, Singapore, 1991).
4. Some more recent articles on DVMs: P. Grosfils, J. B. Boon, and P. Lallemand, Spontaneous fluctuation correlations in thermal lattice-gas automata, *Phys. Rev. Lett.* **68**:1007 (1992); P. Chauvat and R. Gatignol, Euler and Navier–Stokes description for a class of DVMs with different moduli, *TTSP* **21**:417–435 (1992); H. Cabannes, Global solutions of the DVMs with multiple collisions, *Eur. Mech B/Fluids II* **4**:415–437 (1992); *Mech. Res. Comm.* **12**:289–294 (1994); H. Cabannes and Dang Hong Tiem, Exact solutions for a semi-continuous model of the Boltzmann equation, *C. R. Acad. Sci. Paris II* **318**:1583–1590 (1994); H. Cornille, $9v_i$, $15v_i$ DVMs with and without multiple collisions, *TTSP* **24**:709–730 (1995); C. Buet, A discrete-velocity scheme for the Boltzmann operator of rarefied gas-dynamics, *TTSP* **25**:333–360 (1996); V. Panferov, Research report No. 1997-22/ISSN 0347-2809, (1997); I. Nicodin, R. Gatignol, J. P. Duruisseau, and A. d'Almeida, *Int. Symp. Rar. Gas Dyn.* **295**:302 (1998); I. Nicodin, R. Gatignol, and S. Balint, *Int. Conf. Non-Linear Pb. in Aviat. and Aero.* (2000); H. Cornille, Exact solutions for nonconservative DVMs, *TTSP* **29**:141–155 (2000); E. Gabetta, L. Pareschi, and M. Ranconi, Central schemes for hydrodynamical limits of DVMs, *TTSP* **29**:465–477 (2000); S. A. Amossov, Two-level DVMs for binary-mixtures, *TTSP* **31**:125–139 (2002).
5. C. Cercignani, On the thermodynamics of a discrete velocity gas, *TTSP* **23**:1 (1994); (some sentences are reported in the Introduction); Temperature, entropy and kinetic theory, *J. Stat. Phys.* **87**:1097 (1997).
6. C. Cercignani and H. Cornille, Shock waves for a DVMs mixture, *J. Stat. Phys.* **99**:967 (2000); H. Cornille and C. Cercignani, A class of planar DVMs for gas mixtures, *J. Stat. Phys.* **99**:967 (2000); *On a Class of Large Size DVMs for Gas Mixtures*, ECMI2000, G. Muscato, ed. (Springer, 2001), pp. 157–163; On a class of planar DVMs for gas mixtures, in *Wascom99* (World Scientific, Singapore), p. 119; Large size DVMs for gas mixtures, *JPA* **34**:2985 (2001).
7. A. V. Bobylev and C. Cercignani, DVMs without nonphysical invariants, *J. Stat. Phys.* **91**:327 (1998); **97**:677 (1999) (however spurious invariants were found); C. Cercignani and A. V. Bobylev, DVMs for mixtures, *TTSP* **29**:209 (1999).
8. H. Cornille, Large size planar DVMs and two continuous relations, in *Wascom2001* (World Scientific, Singapore, 2002), pp. 169–175, (some partial results for Classes I-II were briefly reported).
9. I. Nicodin, La Modélisation pour des Gas Discrets des phénoménes d'Evaporation/Condensation, Thesis (Paris, 2001).
10. R. Caflisch, Half-space problem for the Boltzmann equation at zero temperature, *Comm. Pure Appl. Math.* **38**:529–547 (1985); C. Cercignani, Half-space problems in the kinetic

theory of gases, in *Trends in Applications of Pure Mathematics to Mechanics* (Springer-Verlag, Berlin, 1986), pp. 35–50; C. Cercignani, R. Illner, M. Pulvirenti, and M. Shinbrot, On nonlinear stationary half-space problems in DVMs, *J. Stat. Phys.* **52**:885–897 (1988); H. Cabannes, Evaporation and condensation problems in DVMs, *Eur. J. Mech. B Fluids* **13**:685–699 (1994); A. d'Almeida and R. Gatignol, Boundary conditions in DVMs and evaporation and condensation problems, *Eur. J. Mech. B Fluids* **16**:401–428 (1997); S. Kawashima and S. Nishibata, Stationary waves for DVMs in the half space with reflective boundaries, *Comm. Math. Phys.* **211**:183–206, (2000); Review: Y. Sone, Kinetic theoretical studies of the half-space problem of evaporation and condensation, *TTSP* **29**:227–259 (2000); H. Cornille and A. d'Almeida, Temperature and pressure criteria for half-space DVMs, *Eur. J. Mech. B Fluids* **13**:355–356 (2002).

11. H. Cornille, Half-space large size DVMs, *Con. Theo. P July 2002, Ann. Inst. H. Poincaré* **4**:741–754 (2003).